Stability of charged rotating black holes for linear scalar perturbations

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Abstract

In this thesis, the stability of the family of subextremal Kerr–Newman spacetimes is studied in the case of linear scalar perturbations.

That is, *nondegenerate energy bounds* (NEB) and *integrated local energy decay* (ILED) results are proved for solutions of the wave equation on the domain of outer communications. The main obstacles to the proof of these results are superradiance, trapping and their interaction. These difficulties are surmounted by localising solutions of the wave equation in phase space and applying the vector field method. Miraculously, as in the Kerr case, superradiance and trapping occur in disjoint regions of phase space and can be dealt with individually.

Trapping is a high frequency obstruction to the proof whereas superradiance occurs at both high and low frequencies. The construction of energy currents for superradiant frequencies gives rise to an unfavourable boundary term. In the high frequency regime, this boundary term is controlled by exploiting the presence of a large parameter. For low superradiant frequencies, no such parameter is available. This difficulty is overcome by proving quantitative versions of mode stability type results. The mode stability result on the real axis is then applied to prove integrated local energy decay for solutions of the wave equation restricted to a bounded frequency regime.

The (ILED) statement is necessarily degenerate due to the trapping effect. This implies that a nondegenerate (ILED) statement must lose differentiability. If one uses an (ILED) result that loses differentiability to prove (NEB), this loss is passed onto the (NEB) statement as well. Here, the geometry of the subextremal Kerr–Newman background is exploited to obtain the (NEB) statement directly from the degenerate (ILED) with no loss of differentiability.

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Statement of Originality

I hereby declare that my thesis entitled "Stability of charged rotating black holes for linear scalar perturbations" is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is concurrently submitted for any such degree of diploma or other qualification.

The work contained in this thesis is original and was entirely performed by myself at the Cambridge Centre for Analysis, University of Cambridge in the period between October 2010 and July 2014.

Contents

Chapter 1

Introduction

1.1 Context

In 1915, after a seven year struggle to incorporate gravity in his theory of relativity, Einstein published [Ein15]. In this pioneering work, he formulated the fundamental equations of the general theory of relativity:

$$\mathbb{G}_{\mu\nu} = \mathbb{T}_{\mu\nu}. \tag{Einstein field equations}$$

The unknown in the theory is a *spacetime* $(\mathfrak{M}, \mathfrak{g})$, where $\mathfrak{M}$ is a Lorentzian manifold with metric $\mathfrak{g}$. The *Einstein tensor* $\mathbb{G}_{\mu\nu}$ describes the curvature of spacetime and the *energy-momentum tensor* $\mathbb{T}_{\mu\nu}$ models the energy and matter within that spacetime. For this system to be fully determined, $\mathbb{T}_{\mu\nu}$ must be specified and equations must be specified for the matter fields.

The Einstein field equations model the gravitational interaction of space, matter and energy. Therefore, the geometry of the spacetime and the matter and energy present are interconnected. In this, the Einstein equations are similar to the Maxwell equations, where charges and currents determine an electromagnetic field.

Einstein's geometrisation of the Newtonian theory of gravity has the immediately remarkable consequence that the theory is nontrivial even in the case of vacuum ($\mathbb{T}_{\mu\nu} = 0$). In this case the Einstein field equations reduce to

$$\mathbb{R}_{\mu\nu} = 0, \tag{Einstein vacuum equations}$$

where $\mathbb{R}_{\mu\nu}$ is the Ricci curvature of $(\mathfrak{M}, \mathfrak{g})$.

The Einstein field equations can be viewed as a system of ten nonlinear partial differential equations for the unknown metric $\mathfrak{g}$. Their analysis is therefore very difficult. In fact, the formulation and proof of well-posedness of the initial value problem for the Einstein vacuum equations was achieved more than thirty years after [Ein15] was published. This was done by Fourès-Bruhat [FB52] and Choquet-Bruhat and Geroch [CBG69] after the ground-breaking works of Friedrichs, Schauder, Sobolev, Petrovsky, Leray and others in the interim.

The identification of explicit solutions (those that can be written in closed form) is a useful first step in understanding a theory in which the fundamental equations are nonlinear. The simplest solution of the vacuum Einstein equations is Minkowski space $(\mathbb{R}^4, diag(-1, 1, 1, 1))$. This is the space in which the special theory of relativity is formulated.

In the early years of the study of general relativity there was considerable interest in deriving and interpreting explicit solutions of the Einstein equations under simplifying assumptions.

In 1916, Schwarzschild discovered a solution of the vacuum Einstein equations which contains a region of spacetime which cannot communicate with the rest of the spacetime [Sch03]. Such regions were later named *black holes* by Wheeler.[1] The discovery of a black hole solution of the Einstein–Maxwell electrovacuum equations (where $\mathbb{T}_{\mu\nu}$ is defined through the Maxwell equations, see (2.1.5)) followed shortly after in [Rei16] and [Nor18]. This charged black hole solution is known as the Reissner–Nordström spacetime.

Both the Schwarzschild and Reissner–Nordström solutions are spherically symmetric. It was only much later that explicit metrics for spacetimes containing rotating black holes were discovered. In 1963, Kerr derived an explicit solution of the vacuum Einstein equations that models a rotating black hole in [Ker63]. In [NCC$^+$65], Newman et al. derived charged rotating black hole solutions of the Einstein–Maxwell electrovacuum equations. These solutions are known as the Kerr–Newman family of spacetimes. The family is parametrised by three physical parameters: the mass M, angular momentum density a and charge Q.[2] *Subextremal* means that $0 \leq a^2 + Q^2 < M^2$. It is the subextremal family (and the extremal case $a^2 + Q^2 = M^2$) of Kerr–Newman spacetimes in which a charged rotating black hole is present. The *fast Kerr–Newman spacetimes* (where $a^2 + Q^2 > M^2$) have profoundly different structure, see [Car73].

Table 1.1 illustrates the relationships between each of the solutions mentioned above as subfamilies of the Kerr–Newman family.

	Uncharged $Q = 0$	**Charged** $Q \neq 0$
Non-rotating $a = 0$	Schwarzschild g_M	Reissner–Nordström $g_{Q,M}$
Rotating $a \neq 0$	Kerr $g_{a,M}$	Kerr–Newman $g_{a,Q,M}$

Table 1.1: The subfamilies of the Kerr–Newman family

The Kerr–Newman solutions are of particular significance in light of the

"No-Hair" Conjecture. *The domain of outer communications of a smooth, stationary, four dimensional, electrovacuum, connected black hole solution is isometrically diffeomorphic to that of a member of the Kerr–Newman family of black holes.*

Conditional versions of this conjecture were proved under the additional assumption of either axisymmetry or analyticity in the work of Bunting, Carter, Hawking, Mazur and Robinson in the 1970s and 80s (see [Heu96] for a detailed account). More recently, conditional versions of the conjecture have been proved under much weaker assumptions in [IK09] and [AIK10] (for the Kerr case) and [Won09] (for the Kerr–Newman case).

The existence of black holes is perhaps the most striking prediction of general relativity.

[1] The interested reader is referred to [DR13] for an excellent account of the intriguing history of the Schwarzschild solution.

[2] The parameter Q represents the total electric charge of the spacetime, see [Wal84, §12.3]

This prediction arises from the study of explicit solutions in the hope that they may be suggestive of the behaviour of general solutions. However, if one wishes to draw any such conclusions from an explicit solution it is imperative to prove that the solution in question is *stable* in an appropriate sense.

This motivates the focus of this thesis: the study of the stability of the subextremal Kerr–Newman family of explicit solutions of the Einstein electrovacuum equations.

1.2 The black hole stability problem

Since the Kerr–Newman solution is thought to be the unique stationary electrovacuum black hole spacetime, the question of its stability is closely related to the plausibility of the concept of a black hole.

The ultimate goal is to understand the dynamical stability of the Kerr–Newman solutions as a family of solutions to the Cauchy problem for the Einstein–Maxwell Equations, affirming the following:

Conjecture (Global stability of Kerr–Newman). *Any small perturbation of the initial data set of a Kerr–Newman spacetime has a global future development with a complete future null infinity which, within its domain of outer communication, behaves asymptotically like a (another) Kerr–Newman solution.*

This is one of the most important unresolved issues in the theory of relativity (see [Kla07] for an insightful discussion of this and other open problems).

1.2.1 Linear stability

The only asymptotically flat spacetime which is known to be globally stable with respect to nonlinear perturbations is Minkowski space. This was first proved by Christodoulou and Klainerman in the monumental [CK93]. Following their philosophy, the first step toward the proof of the nonlinear stability of the Kerr–Newman solution is to understand the behaviour of *scalar perturbations*, i.e. solutions of the linear wave equation

$$\Box_{g_{M,a,Q}}\psi = 0. \tag{1.2.1}$$

This stability problem may be thought of as a "poor-man's version" of the problem of *gravitational perturbations*, obtained by linearising the Einstein equations with respect to a fixed subextremal Kerr–Newman metric $g_{M,a,Q}$.

The particular understanding of (1.2.1) required is a proof that ψ is uniformly bounded and decays (sufficiently rapidly) in time. This *stability with respect to linear scalar per-*

turbations is proved in Chapter 3 of this thesis, see Theorem 3.2.1.[3]

1.2.2 Carter's separation & mode stability

A general subextremal Kerr–Newman metric possesses two Killing fields T and Φ so the wave equation (1.2.1) admits solutions of the form

$$\psi(t,r,\theta,\phi) = e^{-i\omega t}e^{im\phi}\tilde{\psi}(r,\theta), \qquad \text{where } \ \omega \in \mathbb{C}, m \in \mathbb{Z}.$$

Carter discovered in [Car68] that (1.2.1) can be formally separated. The wave equation therefore admits *mode solutions* of the form

$$\psi(t,r,\theta,\phi) = R^{(a\omega)}_{m\ell}(r)S^{(a\omega)}_{m\ell}(\theta)e^{im\phi}e^{-i\omega t}, \text{where } \ \ell \in \mathbb{Z},\ \ell \geq |m|. \tag{1.2.2}$$

The function $S^{(a\omega)}_{m\ell}(\theta)$ solves a Sturm-Liouville problem and $R^{(a\omega)}_{m\ell}(r)$ satisfies the *Carter ODE*:

$$\frac{d^2}{dr^2}R^{(a\omega)}_{m\ell}(r) + \left(\omega^2 - V^{(a\omega)}_{m\ell}(r)\right) R^{(a\omega)}_{m\ell}(r) = 0, \tag{Carter ODE}$$

where $V^{(a\omega)}_{m\ell}(r)$ is a smooth potential.

A priori, (1.2.1) may admit mode solutions that have finite energy but grow exponentially in time, i.e. solutions of the form above with ω in the upper half-plane. The (qualitative) statement that such modes do not exist is known as *mode stability*.

The proof of Theorem 3.2.1 (quantitative boundedness and decay for solutions of (1.2.1)) given in Chapter 3 depends on a quantitative refinement of the qualitative statement of mode stability.

The necessary refinement is proved by first extending the mode stability statement to exclude resonances on the real axis and then refining this qualitative statement to a quantitative estimate.

In Chapter 4 both the qualitative mode stability results (in the upper half-plane and on the real axis) as well as the quantitative estimate are proved for the family of subextremal Kerr–Newman spacetimes. See §4.5 for the precise statements of the mode stability results. The application of the quantitative mode stability results required in the proof of Theorem 3.2.1 is stated as Theorem 4.8.2.

1.3 Main results

In this thesis the stability of the subextremal Kerr–Newman exterior spacetime for linear scalar perturbations is proved. The proof appeals to quantitative mode stability results.

[3]In the Kerr case, the linear stability problem has been resolved in [DR11a, DRSR14], see §1.5.3.

The main results are summarised below.

1.3.1 Linear stability results

The primary goal here is to provide a proof of the energy estimates (NEB) and (ILED) below. Higher order and pointwise decay results are then derived from these key estimates.

Theorem 1.3.1. *Solutions of the wave equation* (1.2.1) *on a subextremal Kerr–Newman exterior spacetime satisfy the following energy estimates:*

- *nondegenerate energy bounds*

$$\int_{\Sigma_\tau} E[\psi] \leq C \int_{\Sigma_0} E[\psi], \tag{NEB}$$

- *integrated local energy decay*

$$\int_0^\tau \int_{\Sigma_s} E_\natural[\psi] + r^{-3-\delta}\psi^2 ds \leq C \int_{\Sigma_0} E[\psi], \tag{ILED}$$

where Σ_0 and Σ_τ are spacelike hypersurfaces and $E[\psi]$ and $E_\natural[\psi]$ are appropriately weighted square sums of the first derivatives of ψ.

The energy $E_\natural[\psi]$ in (ILED) is necessarily degenerate due to the presence of trapped null geodesics (see §1.4). However, trapping is an obstacle to decay but not boundedness. Therefore, (NEB) should not 'see' trapping. That is, $E[\psi]$ in (NEB) is nondegenerate and both sides of the inequality (NEB) contain only first order derivatives of ψ. In this thesis, we extract the fully *nondegenerate* energy bound (NEB) from a *degenerate* integrated local energy statement (ILED) *with no loss of differentiability.* To achieve this, the precise degeneration of (ILED) due to trapping must be understood in phase space.

The precise version of the theorem above is Theorem 3.2.1. Its proof is the content of Chapter 3. An overview of the proof can be found in §3.1.

1.3.2 Mode stability results

In the proof of estimate (ILED), a quantitative energy estimate is required for mode solutions of the form (1.2.2) supported in the bounded frequency range

$$\mathcal{F} \subset \Big\{ (\omega, m, \ell) \in \mathbb{R} \times \{\mathbb{Z} \times \mathbb{Z} \mid \ell \geq |m|\} \;\mid\; \Big(|\omega| + |\omega|^{-1} + |m| + |\ell|\Big) < \infty \Big\}.$$

Theorem 1.3.2. *There exists a constant $C_{\mathcal{F}}$ such that*

$$\int_{\omega\in\mathcal{F}}\sum_{m,\ell\in\mathcal{F}}\left(\left|R^{(a\omega)}_{m\ell}(r_+)\right|^2+\int_{r_0}^{r_1}\left|\partial_r R^{(a\omega)}_{m\ell}\right|^2+\left|R^{(a\omega)}_{m\ell}\right|^2\,dr^*\right)\,d\omega\leq C_{\mathcal{F}}\int_{\Sigma_0}E[\psi], \quad (1.3.1)$$

where each $R^{(a\omega)}_{m\ell}$ solves (Carter ODE) *for $(\omega,m,\ell)\in\mathcal{F}$.*

This estimate is an application of the following quantitative mode stability result

Theorem 1.3.3 (Quantitative mode stability on the real axis)**.** *Let $(\omega,m,\ell)\in\mathcal{F}$. The Wronskian W (given by* (4.3.1)*) satisfies*

$$\sup_{(\omega,m,\ell)\in\mathcal{F}}\left|W^{-1}\right|\leq G(\mathcal{F}).$$

where the function G can, in principle, be given explicitly.

Theorem 1.3.3 provides a quantitative upper bound for $\left|W^{-1}\right|$. This bound implies that any solution of (Carter ODE) can be expressed as a superposition of solutions of (Carter ODE) defined by the asymptotics of $R^{(a\omega)}_{m\ell}$ (see §4.3). This rules out the existence of resonances on the real axis.

Note that it is essential that this estimate on the Wronskian is quantitative in order to prove Theorem 1.3.2.

In proving Theorem 1.3.3, we will also obtain the following qualitative results.

Theorem 1.3.4 (Mode Stability on the real axis)**.** *There exist no non-trivial mode solutions corresponding to $\omega\in\mathbb{R}\setminus\{0\}$.*

Theorem 1.3.5 (Mode Stability)**.** *There exist no non-trivial mode solutions corresponding to $Im(\omega)>0$.*

Chapter 4 contains the proofs of the precise versions of these theorems. An overview of the proof can be found in §4.1.

1.4 The main difficulties

The understanding of *superradiance*, *trapping* and their interaction is crucial in the proof of linear stability given in this thesis. Let us briefly discuss these issues before reviewing the relevant literature.

1.4.1 Future trapped null geodesics

There exist *trapped* null geodesics on a subextremal Kerr–Newman spacetime. These are geodesics that remain for all affine time on, or asymptote in the future to, a constant r value. Physically, this means that photons following such geodesics are neither scattered to infinity, nor do they fall into the black hole. This forces any useful energy current to degenerate in view of a general result of Sbierski [Sbi13], in the spirit of the classical [Ral69]. Here degeneration means that any integrated local energy decay estimate (such as (ILED)) will either have a region in which we do not control all derivatives, or the estimate must lose differentiability. This degeneration is simple on a Schwarzschild spacetime, but considerably more complicated on a subextremal Kerr–Newman spacetime. In fact, the structure of this set can only be completely understood in phase space.

1.4.2 Superradiance

One of the key features of the Kerr–Newman geometry is the existence of an *ergoregion.* This is a region in which the Killing vector field T is spacelike. The presence of the ergoregion complicates the analysis of wave equations as it corresponds to a region in which the conserved energy associated to T is not positive. Thus the conservation law for this energy does not yield control of the solution ψ. This leads to the possibility that the energy flux to null infinity may be larger than the initial energy, hence the term *superradiance.* This phenomenon was first discussed by Zel'Dovich in [Zel71].

1.4.3 Interaction of trapping and superradiance

The main difficulty in proving the required energy estimates in the full subextremal range of Kerr–Newman spacetimes is the interaction of trapping and superradiance. In physical space, it appears that these phenomena must be dealt with simultaneously, as there exist future trapped null geodesics inside the ergoregion. Dealing with the possibility of this interaction thus involves overcoming a potentially serious obstacle. Miraculously, it turns out that superradiance and trapping can be dealt with separately in phase space, see §1.5 for more on this.

1.4.4 Low frequency obstructions

Trapping is a high frequency phenomenon whereas superradiance occurs at low frequencies as well. For superradiant frequencies, the frequency localised energy currents available generate a boundary term with an unfavourable sign. For the superradiant frequencies in the high frequency regime, a large parameter is exploited to deal with this boundary term. In the low frequency regime no large parameter is available, making it difficult to obtain quantitative estimates directly. This low frequency obstruction is overcome through appeal to the estimate of Theorem 1.3.3, see §3.6.

1.5 Historical overview

1.5.1 Classical analysis

In the classical analysis of the stability of black holes, only mode solutions were studied. This work was initiated in the Schwarzschild case in [RW57] and the literature has become vast, see [DR10a] for an overview and references. The mode stability of the Schwarzschild family – the statement that there are no mode solutions of $\Box_{g_M}\psi = 0$ with finite energy at $t = 0$ and $Im(\omega) > 0$ – follows immediately from the observation that the potential V is non-negative in the Schwarzschild case. It is a remarkable fact that this result carries over to gravitational perturbations of Schwarzschild [Vis70] .

The existence of mode solutions on a Schwarzschild background is a consequence of the dimension of the Lie algebra of symmetries of that spacetime (the stationary Killing field T and the rotations). As mentioned in §1.2.2, Carter discovered in [Car68] that, despite the Kerr(–Newman) family possessing only two Killing vector fields, the wave equation can be formally separated on these backgrounds. This is related to the integrability of the geodesic equations on Kerr–Newman spacetimes. This separation was later found to originate from the existence of a "hidden symmetry" in the Kerr–Newman metric, see [WP70] for more on this.

Mode analysis of the Kerr–Newman spacetime reveals that superradiance is frequency specific in the sense that the energy flux through the horizon is negative precisely in the frequency range

$$0 \leq m\omega < \frac{am^2}{2M(M + \sqrt{M^2 - a^2 - Q^2}) - Q^2}.$$

A priori, superradiance may allow for the existence of mode solutions of $\Box_{g_{M,a,Q}}\psi = 0$ with finite initial energy and $Im(\omega) > 0$. As mentioned before, such solutions grow exponentially in time. In the celebrated [Whi89], Whiting showed that no such solutions exist in the Kerr case. Whiting's proof of the mode stability of Kerr is seen today as the culmination of the classical mode analysis. However, for Kerr–Newman spacetimes, the analogue of Whiting's mode stability is absent in the literature.

1.5.2 Limitations of the classical analysis

Classical analysis of mode solutions alone is not enough to resolve the question of linear stability of a spacetime. Recall from §1.2.1 that linear stability refers to solutions of (1.2.1) being uniformly bounded and decaying (sufficiently rapidly) in time. However, mode stability is still completely consistent with general solutions of (1.2.1) with finite initial energy growing in time without bound. Indeed, it is not a priori apparent that general solutions of (1.2.1) can be represented as a superposition of modes with $\omega \in \mathbb{R}$. Even with this established, statements about individual modes do not carry over to the superposition of infinitely many modes without some additional knowledge.

It is only through the estimates of energy-type quantities that one can bound solutions of (1.2.1) and rule out exponentially growing solutions. The modern PDE theory provides powerful tools for the derivation of such energy estimates.

1.5.3 Modern analysis

Boundedness on Schwarzschild

The study of black hole stability from the point of view of modern PDE theory was effectively initiated by the celebrated results of Wald [Wal79] and Kay–Wald [KW87]:

Theorem 1.5.1 (Kay–Wald)**.** *Solutions of the wave equation on a Schwarzschild background arising from sufficiently regular initial data are pointwise uniformly bounded in the exterior region up to and including the horizon.*

The proof of the Kay–Wald Theorem makes no appeal to mode analysis. Instead, the fundamental statements are L^2 based estimates of derivatives of the solution of the wave equation. The pointwise statement follows from commuting the wave equation with certain vector fields and applying Sobolev inequalities. These modern arguments are essential when working with nonlinear equations, such as the Einstein equations. Indeed, it was by these methods that Christodoulou and Klainerman proved the nonlinear stability of Minkowski space in [CK93] (though boundedness alone does not suffice, see below).

From the perspective of modern PDE theory, the proof of the Kay–Wald Theorem away from the horizon follows from a standard application of the energy method (since the vector field T is timelike there). At the horizon T becomes null and the associated conserved energy degenerates. This obstacle was overcome in [Wal79] and [KW87] to obtain estimates up to and including the horizon. However, the geometric arguments employed in this proof are extremely particular to the Schwarzschild solution and as such are very delicate with respect to metric perturbations, see [DR13, §3] for further discussion.

The work of Dafermos–Rodnianski in the slowly rotating Kerr case [DR11b] yields a simpler proof of the Kay–Wald Theorem that does not appeal to fragile geometric properties of Schwarzschild. The Dafermos–Rodnianski approach also highlights the celebrated *red-shift effect* as the physical origin of the boundedness of the horizon energy flux [DR09]. The red-shift effect is discussed further below and in §2.2.3.

Decay on Schwarzschild

The boundedness statement of the Kay–Wald Theorem alone is not sufficient as a linear stability result. Quantitative decay bounds on the solution of the wave equation are also required, where the rate depends only on the size of the initial data. Decay provides a physical interpretation of the linear stability result but moreover, it is the only known mechanism for nonlinear stability (see [CK93, LR10]). It is therefore essential to prove decay estimates in the context of the nonlinear stability problem.

Decay results on Schwarzschild have been proved by many authors [BS03, DR05, BS06a, BS06b, DR07, DR09, Luk10, MMTT10]. These results have led to a better understanding of how geometric aspects of black hole spacetimes interact in the analysis of the wave equation. In particular, the obstruction to decay due to trapping is captured by use of *virial-type* energy currents that degenerate precisely on the trapped geodesics. The use of virial-type estimates originates in the work of Morawetz (c.f. [Mor68]).

The final proof of boundedness and decay in the Schwarzschild case, found in [DR11b] and [DR07] respectively, both take place entirely in physical space and do not require any mode analysis.

Slowly rotating Kerr

In the Kerr case, the linear stability problem for scalar perturbations has been resolved as a result of the work by several authors. The early work in this direction was restricted to perturbations of Schwarzschild and the slowly rotating Kerr case $|a| \ll M$.

In [DR11b], a proof of (NEB) was given for solutions of (1.2.1) on a class of metrics which includes very slowly rotating and small charge Kerr–Newman, $a^2 + Q^2 \ll M^2$, as a special case. The decay result (ILED) in the very slowly rotating Kerr case was then proved in [DR10a]. These results make essential use of the smallness of the parameters a and Q to control the strength of superradiance. In particular, by taking the parameters small enough, the ergoregion can be contained in a region arbitrarily close to the horizon. Energy estimates in the ergoregion can then by obtained from the red-shift estimate (see §2.2.3). Furthermore, the trapping region is bounded away from the horizon, so the difficulties of superradiance and trapping decouple in this case.

Energy decay results were also proved independently in [TT11] and [AB09]. These results also exploit the smallness of the parameters a and Q in a crucial way.

The full subextremal range

The proof of stability for linear scalar perturbations for the full subextremal Kerr family, $|a| < M$, has been achieved recently in [DR11a] and [DRSR14].

The main difficulties of superradiance, trapping and their interaction were overcome by employing a hybrid of the vector field method and classical mode analysis. These two approaches were united by revisiting Carter's separation of the wave equation [Car68] and constructing frequency-localised energy currents. The success of this approach hinges on the miraculous fact brought to light in [DR11a] – superradiance and trapping occur in *disjoint regions of phase space* for the full subextremal range of Kerr spacetimes. The deeper origin of this decoupling (if there is one) remains mysterious and may have bearing on the Kerr(–Newman) uniqueness problem [DR11a, §7.3].

It is possible to partition phase space in a way that allows for bespoke energy estimates to be derived by exploiting the character of each phase space regime. In this way, the degeneration of the estimates due to trapping can be precisely captured.

To overcome superradiance, a frequency localised energy estimate can be derived in the high frequency regime. As mentioned in §1.4.4, this approach is difficult in the low frequency regime. This low frequency obstruction is overcome by appeal to a quantitative refinement of Whiting's celebrated [Whi89]. The necessary refinement was proved very recently by Shlapentokh-Rothman in [SR13] by first extending [Whi89] to exclude resonances on the real axis and then upgrading this qualitative statement to a quantitative estimate.

Once frequency localised estimates are proved in each phase space regime, they are summed up and inverse Fourier transformed to obtain (ILED).

In order to apply Carter's separation, it is necessary to Fourier transform in time. However, there is no a priori guarantee that solutions of the wave equation on a Kerr background are sufficiently integrable to allow this. This final hurdle is leapt over by employing a continuity argument. The separation is carried out on a class of solutions of the wave equation that are assumed to be sufficiently integrable. It is then proved that all solutions of the wave equation on a subextremal Kerr background lie in this class of solutions. This continuity argument is due to [DRSR14]. The argument is simplified by the discovery that for solutions of the wave equation supported only on a single fixed azimuthal frequency, trapping occurs outside the ergoregion.

The (NEB) statement is then extracted from the (ILED) statement to complete the proof of stability of the subextremal family of Kerr spacetimes for linear scalar perturbations.

Turning now to the Kerr–Newman spacetimes, it turns out that all the structure necessary to carry out the strategy of [DR11a] and [DRSR14] carries over from the Kerr to the Kerr–Newman case. A thorough discussion of this structure is given in §3.1 and §4.1.

In [Civ14b] (Chapter 3 of this thesis) the linear stability problem for scalar perturbations for the Kerr–Newman family of spacetimes is resolved.

The argument in [Civ14b] requires the quantitative mode stability result analogous to [SR13]. However, even the analogue of Whiting's mode stability result is absent in the literature for the Kerr–Newman spacetimes. In [Civ14a] (Chapter 4 of this thesis), both the qualitative mode stability results (in the upper half-plane and on the real axis) as well as the quantitative estimate in the spirit of [SR13] are proved.

The extremal and cosmological cases

In the extremal case, the stabilising mechanism of the red-shift effect degenerates and one expects blow-up rather than decay results. Great progress in this direction has been made by Aretakis, see [Are12a, Are12b, Are13a, Are13b]. See [Sch13] for an overview of the $\Lambda > 0$ case. For the $\Lambda < 0$ case, see for example [HS13a], [HS13b] and [HS13c].

Chapter 2

The Kerr–Newman family of spacetimes

2.1 The Kerr–Newman family

We begin with a brief review of the relevant geometric and physical features of the Kerr–Newman spacetimes. For an in-depth treatment, the reader is referred to [Wal84].

The Kerr–Newman metric depends on three physical parameters: the mass M, angular momentum density a and charge density Q. We express these parameters in "natural units" , setting both the gravitational constant G and speed of light c to unity.

Here we consider the *subextremal* family of Kerr–Newman spacetimes in which a charged, rotating black hole is present. Recall that subextremal means that $0 \leq a^2+Q^2 < M^2$.

We first fix the underlying manifold we wish to consider, then discuss the Kerr–Newman metric in suitable local coordinates in §2.1.2.

2.1.1 The underlying manifold

We first make a precise definition of the manifold we wish to consider. Let

$$\mathcal{M} = \{t^* > -\infty,\ y^* \geq 0,\ \theta^* \in [0,\pi],\ \phi^* \in [0,2\pi]\}\,.$$

Here θ^* and ϕ^* are standard spherical coordinates on $\mathbb{S}^2$. This is a manifold with boundary $\partial\mathcal{M} = \mathcal{H}^+ = \{y^* = 0\}$. This boundary is the event horizon. We also define the vector fields $T = \partial_{t^*}$ and $\Phi = \partial_{\phi^*}$ and denote the one parameter family of transformations generated by T by φ_τ.

We will define a family of metrics on $\mathcal{M}$, parametrised by M, a and Q. In §3.7.4 we will be concerned with the smooth dependence of this family on the parameters. The precise dependence we require is given in Lemma 2.2.1.

Before defining this family of metrics, it is convenient to define coordinate systems that depend on the parameters M, a and Q.

Kerr–Newman star coordinates

We introduce the Kerr–Newman star coordinate chart (t^*, r, θ, ϕ^*), which depends on the parameters $a^2+Q^2 < M^2$. For each triple $a^2+Q^2 < M^2$, set $r_\pm = M \pm \sqrt{M^2-a^2-Q^2}$. Then define a new coordinate r, depending smoothly on y^* and the parameters and such that $r = r_+$ on $\mathcal{H}^+$. We denote by Z^* the smooth extension of the Kerr–Newman star coordinate vector field ∂_r to $\mathcal{M}$.

It is often convenient to work with a rescaled version of r, denoted by r^* and defined only in the interior of $\mathcal{M}$ by

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta}, \qquad r^*(3M) = 0, \tag{2.1.1}$$

where

$$\Delta := (r-r_+)(r-r_-) = r^2 - 2Mr + a^2 + Q^2. \tag{2.1.2}$$

Note that Δ vanishes on $\mathcal{H}^+$ and that the range $\{r > r_+\}$ corresponds to $\{r^* > -\infty\}$. In star coordinates $T = \partial_{t^*}$ and $\Phi = \partial_{\phi^*}$.

Remark For subextremal Kerr–Newman metrics, $0 < a^2 + Q^2 < M^2$, hence

$$0 < r_\pm = M \pm \sqrt{M^2 - a^2 - Q^2} < 2M.$$

This humble pair of inequalities is crucial to many of the arguments of this thesis.

Boyer–Lindquist coordinates

We define the Boyer-Lindquist coordinates (t, r, θ, ϕ) by applying the coordinate transformations

$$\begin{cases} t = t^* - \bar{t}(r), & d\bar{t}(r) = \frac{r^2+a^2}{\Delta^2}, \\ \text{and} \quad \phi = \phi^* - \bar{\phi}(r), & d\bar{\phi}(r) = \frac{a}{\Delta}. \end{cases} \tag{2.1.3}$$

See [DR10a] for the details and explicit definitions of $\bar{t}(r)$ and $\bar{\phi}$. We will denote the Boyer–Lindquist ∂_r by Z_{BL}. It will turn out that Z_{BL} defines the directional derivative that does not degenerate in the integrated decay estimate due to trapping (see §1.4.1 and (3.5.11)) but it is Z^* which is regular at the horizon. We therefore define the following combination of Z^* and Z_{BL}.

Definition 2.1.1. *For each $a^2 + Q^2 < M^2$, let $\chi(r)$ be a cut-off function such that $\chi = 1$ for $r \geq r_\natural$ and $\chi = 0$ for $r \leq (r_+ + r_\natural)/2$, where $r_\natural$ is sufficiently close to r_+. Finally, for each $a^2 + Q^2 < M^2$, we define the vector field*

$$Z = \chi Z_{BL} + (1-\chi)Z^*.$$

2.1.2 The Kerr–Newman metric

With M, a and Q as above, we finally define a Kerr–Newman metric on the interior of $\mathcal{M}$ in Boyer–Lindquist coordinates by

$$\begin{aligned} g_{M,a,Q} &:= -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\Big(adt - (r^2+a^2)d\phi\Big)^2 \\ & \quad +\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \qquad (2.1.4) \\ \text{where } \rho^2 &= r^2 + a^2\cos^2\theta \end{aligned}$$

and Δ is defined by (2.1.2). Applying (2.1.3) in reverse, we see indeed that the metric extends regularly to $\mathcal{H}^+$. Note that $\mathcal{H}^+$ is a null hypersurface.

The metric $g_{M,a,Q}$ together with appropriate $F_{\mu\nu}$ satisfies the following system of partial differential equations, known as the *electrovacuum Einstein–Maxwell equations*:

$$\begin{cases} R_{\mu\nu} &= 2\left(F_{\mu\beta}F_{\nu}{}^{\beta} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right), \\ \nabla\cdot F &= 0 \\ \text{and } \ dF &= \nabla_\alpha F_{\beta\gamma} + \nabla_\gamma F_{\alpha\beta} + \nabla_\beta F_{\gamma\alpha} = 0, \end{cases} \qquad (2.1.5)$$

where $R_{\mu\nu}$ is the Ricci curvature of $\mathcal{M}$. We call the tensor $F_{\mu\nu}$ the *electromagnetic tensor*. The interested reader is referred to [Wal84, p. 313] for more details.

Remark The Kerr–Newman family has three well-known subfamilies. When $Q = 0$, (2.1.4) is the Kerr metric. Letting $a = 0$, we obtain the Reissner-Nordström metric. Finally, if we set $a = Q = 0$ we simply have the Schwarzschild metric. See Table 1.1.

The vector fields T and Φ are Killing and if $a \neq 0$ they span the Lie algebra of Killing fields. In Boyer–Lindquist coordinates we have $T = \partial_t$ and $\Phi = \partial_\phi$. Thus it is clear from (2.1.4) that $\mathcal{L}_T g = \mathcal{L}_\Phi g = 0$. T is referred to as the *stationary* Killing field and Φ is called the *axisymmetric* Killing field. As $r \to \infty$, T is asymptotically future pointing timelike and Φ is asymptotically orthogonal to T.

The determinant of the metric is simply $\det(g_{\mu\nu}) = -\rho^4\sin^2\theta$. Hence the volume form on a $(3+1)$ dimensional Kerr–Newman manifold in Boyer–Lindquist coordinates is

$$dV = \rho^2\ dt\ dr\ dV_{\mathbb{S}^2} = \rho^2\sin\theta\ dt\ dr\ d\theta\ d\phi. \qquad (2.1.6)$$

2.1.3 The wave equation

As mentioned in the introduction, the first step in the journey toward resolution of the nonlinear stability problem is the analysis of the linear stability problem for scalar perturbations, using sufficiently robust techniques. The simplest such linear problem is the scalar wave equation

$$\Box_g \psi = 0,$$

which may be thought of as a "poor-man's version" of the linearised Einstein equations (taken around a subextremal Kerr–Newman metric). Thus the boundedness and energy decay of such ψ on a Kerr–Newman background may be thought of as stability of this spacetime for linear scalar perturbations.

The wave equation on a Lorentzian manifold can be written in coordinates as

$$\Box_g \psi = \frac{1}{\sqrt{-g}} \partial_\alpha \left(\sqrt{-g} g^{\alpha\beta} \partial_\beta \psi \right) = 0. \tag{2.1.7}$$

For the Kerr–Newman metric in Boyer–Lindquist coordinates, this is

$$\begin{aligned} \frac{1}{\rho^2 \sin\theta} \Bigg[\left(a^2 \sin^2\theta - \frac{(a^2+r^2)^2}{\Delta} \right) \partial_t^2 \psi - \frac{a^2}{\Delta} \partial_\phi^2 \psi \\ - \frac{2a(2Mr - Q^2)}{\Delta} \partial_t \partial_\phi \psi + \partial_r (\Delta \partial_r \psi) + \not{\Delta}_{\mathbb{S}^2} \psi \Bigg] = 0, \end{aligned} \tag{2.1.8}$$

where $\not{\Delta}_{\mathbb{S}^2}$ denotes the (unit) spherical Laplacian:

$$\not{\Delta}_{\mathbb{S}^2} \psi = \frac{1}{\sin\theta} \partial_\theta (\sin\theta \partial_\theta \psi) + \frac{1}{\sin^2\theta} \partial_\phi^2 \psi.$$

Similarly, we denote the covariant derivative on the unit sphere by $\not{\nabla}_{\mathbb{S}^2}$ and the gradient

$$\not{\nabla}_{\mathbb{S}^2} \psi = \frac{\partial \psi}{\partial \theta} \partial_\theta + \frac{1}{\sin\theta} \frac{\partial \psi}{\partial \phi} \partial_\phi.$$

We also denote

$$|\not{\nabla}_{\mathbb{S}^2} \psi|^2 := (\partial_\theta \psi)^2 + \frac{1}{\sin^2\theta} (\partial_\phi \psi)^2.$$

We introduce the related operators

$$\not{\Delta} \psi = \frac{1}{\rho^2} \not{\Delta}_{\mathbb{S}^2} \psi \qquad \text{and} \qquad \not{\nabla} \psi = \frac{1}{\rho} \not{\nabla}_{\mathbb{S}^2} \psi.$$

Note that $|\not{\nabla} \psi|^2 = \dfrac{1}{\rho^2} |\not{\nabla}_{\mathbb{S}^2} \psi|^2$.

Carter discovered in [Car68] that (2.1.8) can be formally separated. The separation introduces frequencies ω, m, and λ. This provides us with the means to frequency localise and thus capture frequency specific phenomena. This separation is carried out in §3.3.3.

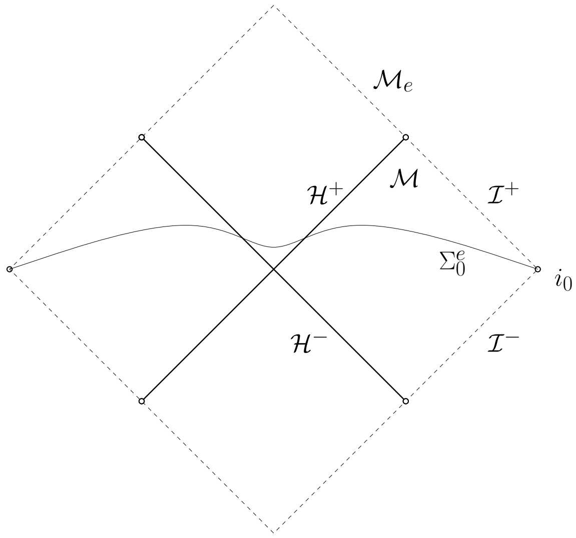

Figure 2.1.1: The maximal globally hyperbolic extension

2.1.4 The maximal globally hyperbolic extension

In the physical application, the asymptotically flat spacetime $(\mathcal{M}, g_{M_a,Q})$ is meant to represent the gravitational field in the vicinity of an isolated charged rotating black hole.

From a purely mathematical perspective, there is a larger manifold that we could consider. By appropriate coordinate changes and combining coordinate patches one can construct the *maximally extended Kerr–Newman manifold* $\mathcal{M}^{\mathcal{E}}$, see [Car73] and [HE73]. However, $\mathcal{M}^{\mathcal{E}}$ is not compatible with the dynamical formulation of general relativity. If we consider the spacetime as the solution of an initial value problem with data prescribed on a Cauchy hypersurface, $\mathcal{M}^{\mathcal{E}}$ will necessarily contain inextendable causal geodesics which do not intersect that Cauchy hypersurface. Indeed, the maximally extended Kerr–Newman solutions are quit bizarre – in particular, they contain closed timelike curves.

In the dynamical formulation of the Einstein equations, the correct manifold to consider is the Cauchy development of initial data prescribed on a suitable hypersurface. In the maximally extended Kerr–Newman spacetime $\mathcal{M}^{\mathcal{E}}$, there are two regions which are isometric to the original exterior region $\mathcal{M}$. This suggests that the Cauchy hypersurface used in the initial value problem should have have topology $\mathbb{S}^2 \times \mathbb{R}$ with two asymptotically flat ends. Let Σ_0^e be such a hypersurface. Viewing the Einstein–Maxwell equations as a hyperbolic system of PDE, the Kerr–Newman manifold is then the solution of (2.1.5) with appropriate initial data prescribed on Σ_0^e, see [Wal84]. We will denote this solution by $(\mathcal{M}_e, g^e_{M,a,Q})$.

The manifold $\mathcal{M}$ that we have considered thus far is a submanifold of the Cauchy development $\mathcal{M}_e$ of Σ_0^e with $g^e_{M,a,Q} = g_{M,a,Q}$ on $\mathcal{M}$.

The Penrose diagram of $\mathcal{M}_e$, along the axis of symmetry, is depicted in Figure 2.1.1.

The boundary component $\mathcal{I}^+$ is known as *future null infinity* and comprises the limit points of future directed null rays in $\mathcal{M}$ along which $r \to \infty$. Similarly, $\mathcal{I}^-$ comprises the limit points of past directed null rays for which $r \to \infty$. We call $\mathcal{I}^-$ *past null infinity.* The remaining boundary components i^0 and $i^\pm$ are called *spacelike infinity* and *future (past) timelike infinity*, respectively. In the physical application, $\mathcal{I}^+$ is an idealisation of far away astrophysical observers receiving radiation from the system.

The maximal globally hyperbolic development $\mathcal{M}_e$ will be useful in the proof of (NEB), see §3.8.2.

2.2 Preliminaries

The main results of this thesis are energy estimates for solutions ψ of the wave equation (2.1.8). Here energy refers to an integral of square sum of derivatives of ψ of the form

$$\int_\Sigma \sum_{1\leq i_1+i_2+i_3\leq j} \left|\nabla\!\!\!\!/^{\,i_1} T^{i_2} Z^{i_3}\psi\right|^2 \, dg_\Sigma,$$

where Σ is a spacelike hypersurface. We typically prove estimates in the case $j = 1$ and then extend the results to the higher order case $j \geq 1$ as corollaries.

2.2.1 Foliation and well-posedness

In order to formulate the initial value problem for the wave equation, we must prescribe data on a suitable hypersurface. The submanifold $\{t^* \geq 0\}$ of $\mathcal{M}$ is the future Cauchy development of $\Sigma_0 = \{t^* = 0\}$. We are interested in the behaviour of solutions of the wave equation (2.1.8) in the future of Σ_0. To prove energy estimates, we use a foliation of the type described in [DR11a, §4]. Letting φ_τ denote the $1-$parameter family of diffeomorphisms generated by the vector field T, we define the hypersurfaces

$$\Sigma_\tau = \varphi_\tau(\Sigma_0) = \{t^* = \tau\}\,.$$

Note that each leaf of this foliation is terminates on $\mathcal{H}^+$ and spatial infinity i^0. (More details can be found in [DR11a, §2].) Let $\tau_2 > \tau_1$ so that Σ_{τ_2} lies in the future of Σ_{τ_1}. Denote the region bounded by Σ_{τ_1}, Σ_{τ_2} and $\mathcal{H}^+$ by

$$\mathcal{R}(\tau_1,\tau_2) = \bigcup_{\tau_1\leq\tau\leq\tau_2} \Sigma_\tau$$

so that the future Cauchy development of Σ_0 is precisely

$$\mathfrak{D}^+(\Sigma_0) = \bigcup_{\tau' \geq \tau} \mathcal{R}(0, \tau').$$

The lapse function associated with the foliation is uniformly bounded above and away from zero. That is, there exist positive uniform constants $b < B$ such that for all $\tau \geq 0$,

$$B \geq (-g_{M,a,Q}(\nabla\tau, \nabla\tau))^{-\frac{1}{2}} \geq b > 0. \tag{2.2.1}$$

In particular this means that each Σ_τ is spacelike.

By the smooth coarea formula, for a smooth, integrable function F, there exist constants c and C such that

$$c \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma_\tau} F \right) d\tau \leq \int_{\mathcal{R}(\tau_1,\tau_2)} F \leq C \int_{\tau_1}^{\tau_2} \left(\int_{\Sigma_\tau} F \right) d\tau. \tag{2.2.2}$$

This is used many times without further comment in what follows.

We denote the causal future and past (restricted to $\mathcal{M}$) of a set $A \subset \mathcal{M}$ by $J^+(A)$ and $J^-(A)$ respectively.

Proposition 2.2.1. *[Hör07, Theorem 23.2.4] Let Σ_0 be as above and let n_Σ be the (future directed) unit normal to Σ_0. For any*

$$\psi|_{\Sigma_0} = \psi_0 \in H^k_{loc}(\Sigma_0) \quad \text{and} \quad n_{\Sigma_0}\psi = \psi_1 \in H^{k-1}_{loc}(\Sigma_0), \quad k \geq 1,$$

there exists a unique solution to the initial value problem

$$\begin{cases} \Box_g \psi = 0, \\ \psi|_{\Sigma_0} = \psi_0, \\ n_{\Sigma_0}\psi = \psi_1, \end{cases} \tag{2.2.3}$$

such that

$$\psi(\tau, \cdot) \in C([0,\infty); H^k_{loc}(\Sigma_\tau)) \cap C^1([0,\infty); H^{k-1}_{loc}(\Sigma_\tau)).$$

Furthermore, the solution depends smoothly on the parameters a and Q. The precise dependence on Q is as follows.

Lemma 2.2.1. *Let $Q^2 < M^2 - a^2$ and $\{Q_k\}_{k=1}^\infty$ have the limit $Q_k \to Q$. Define the* *sequence* $\{\psi_k\}_{k=1}^{\infty}$ *as the solutions of*

$$\begin{cases} \Box_{g_{M,a,Q_k}}\psi_k = 0, \\ \psi_k|_{\Sigma_0} = \psi_0, \\ n_{\Sigma_0}\psi = \psi_1, \end{cases}$$

where ψ_0 *and* ψ_1 *are as in Proposition 2.2.1. Then, for every* $j \geq 1$ *and* $\tau \geq 0$*,*

$$\lim_{k\to\infty}\int_{\Sigma_\tau}\sum_{1\leq i_1+i_2+i_3\leq j}\left|\nabla\!\!\!\!/^{\,i_1}T^{i_2}Z^{i_3}\psi_k\right|^2 dg_{\Sigma_\tau}^{(k)} = \int_{\Sigma_\tau}\sum_{1\leq i_1+i_2+i_3\leq j}\left|\nabla\!\!\!\!/^{\,i_1}T^{i_2}Z^{i_3}\psi\right|^2 dg_{\Sigma_\tau}.$$

Note that in the expression above, the geometric objects $\nabla\!\!\!\!/$, Z, and the volume form $dg_{\Sigma_\tau}^{(k)}$ depend on the metric g_{M,a,Q_k}.

The analogous statement holds for a dependence, see [DRSR14, Lemma 4.1.1].

In §3.7.4, we will make explicit use of the smooth dependence of the solution on Q.

2.2.2 The sign of a

Let ψ be a solution of $\Box_{g_{M,a,Q}}\psi = 0$, for some M, a, Q. Then defining

$$\tilde{\psi}(y^*, t^*, \theta^*, \phi^*) = \psi(y^*, t^*, \theta^*, 2\pi - \phi^*),$$

we have that $\tilde{\psi}$ satisfies $\Box_{g_{M,-a,Q}}\tilde{\psi} = 0$.

Taking all objects and quantities defined with respect to the metric $g_{M,-a,Q}$, the results of §3.2 for $\tilde{\psi}$ are equivalent to those for ψ with respect to $g_{M,a,Q}$. Therefore, it suffices to consider $a \geq 0$.

This reduction is of no conceptual significance and is made only to simplify the notation when discussing the superradiant frequency range, see (3.3.21) and (3.3.22).

The reader can assume that $a \geq 0$ everywhere in this thesis, though it is only strictly necessary for statements that refer explicitly to frequency-dependent functions.

2.2.3 Energy currents

The vector field method

The *vector field method* is a robust technique for deriving L^2-based identities with the help of geometrically natural vector fields. These are in turn used to link the geometry of the spacetime $(\mathcal{M}, g)$ to the behaviour of solutions of $\Box_g\psi = 0$ on $\mathcal{M}$. There are two aspects to this method. Firstly, that of *vector field multipliers* which are used extensively in this thesis. Secondly, by commuting the wave equation with certain vector fields one can derive higher order and pointwise estimates. For example, commutation vector fields are used extensively in proving Corollary 3.2.4 below. A brief history of the vector field method can be found in [Kla10].[1]

The vector field multiplier method is based on applying the Divergence Theorem to a particular class of 1-forms, known as *energy currents*. Energy currents are constructed from the *energy momentum tensor*. The energy momentum tensor associated with the wave equation (2.1.7) on a Lorentzian manifold is given by

$$\mathbb{T}_{\mu\nu}[\psi] = \partial_\mu\psi\partial_\nu\psi - \frac{1}{2}(g^{\alpha\beta}\partial_\alpha\psi\partial_\beta\psi)g_{\mu\nu}.$$

Given a vector field V and a (sufficiently regular) function ψ, we define the following energy currents.

$$\begin{aligned} \mathbb{J}^V_\nu[\psi] &= \mathbb{T}_{\mu\nu}[\psi]V^\nu, \\ \mathbb{K}^V[\psi] &= \mathbb{T}_{\mu\nu}[\psi]\nabla^\mu V^\nu \\ \text{and } \mathcal{E}^V[\psi] &= \operatorname{div}(\mathbb{T})V = (\Box\psi)d\psi(V). \end{aligned}$$

Note that

$$\nabla^\mu\mathbb{J}^V_\mu[\psi] = \mathbb{K}^V[\psi] + \mathcal{E}^V[\psi].$$

The wave equation is satisfied if and only if the divergence of the energy momentum tensor vanishes. If V is a Killing vector field then $\mathbb{K}^V[\psi] = 0$. These facts are vital in the construction which follows.

We will be working with an inhomogeneous wave equation $\Box_g\psi = F$ due to the necessity of cutting off in time (see §3.3.3). As such, the divergence of the associated energy momentum tensor will not vanish. Thus

$$\mathcal{E}^V[\psi] = FV^\nu\partial_\nu\psi.$$

It will be useful to augment $\mathbb{J}^V_\nu[\psi]$ with a (sufficiently regular) function w. We define

$$\begin{aligned} \mathbb{J}^{V,w}_\nu[\psi] &= \mathbb{J}^V_\nu[\psi] + \frac{1}{8}w\partial_\mu(\psi^2) - \frac{1}{8}(\partial_\mu w)(\psi^2). \\ \text{Hence } \mathbb{K}^{V,w}[\psi] &= \mathbb{K}^V[\psi] - \frac{1}{8}(\Box w)(\psi^2) + \frac{1}{4}w\nabla^\alpha\psi\nabla_\alpha\psi \\ \text{and } \mathcal{E}^{V,w}[\psi] &= \mathcal{E}^V[\psi] + \frac{1}{4}(w\psi)(\Box\psi). \end{aligned}$$

The vector field method essentially refers to applying the Divergence Theorem to $\mathbb{J}^{V,w}_\nu[\psi]$ within a region such as $\mathcal{R}(\tau_1, \tau_2)$ for carefully chosen V and w, to obtain the associated

[1] Available online at http://web.math.princeton.edu/~seri/homepage/papers/John2010.pdf.

energy identity.

$$\int_{\Sigma_{\tau_2}} \mathbb{J}^{V,w}_\mu[\psi] n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}^+\cap\mathcal{R}(\tau_1,\tau_2)} \mathbb{J}^{V,w}_\mu[\psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{R}(\tau_1,\tau_2)} \mathbb{K}^{V,w}[\psi] + \int_{\mathcal{R}(\tau_1,\tau_2)} \mathcal{E}^{V,w}[\psi] = \int_{\Sigma_{\tau_1}} \mathbb{J}^{V,w}_\mu[\psi] n^\mu_{\Sigma_0}.$$

In implementing the vector field method, it is often useful to arrange for the boundary terms to have a "good" sign and treat the bulk terms as error terms. For example, if V is a Killing field and ψ satisfies the wave equation, then the bulk terms vanish and we obtain the following conservation law

$$\int_{\Sigma_{\tau_2}} \mathbb{J}^{V}_\mu[\psi] n^\mu_{\Sigma_\tau} + \int_{\mathcal{H}^+\cap\mathcal{R}(\tau_1,\tau_2)} \mathbb{J}^{V}_\mu[\psi] n^\mu_{\mathcal{H}^+} = \int_{\Sigma_{\tau_1}} \mathbb{J}^{V}_\mu[\psi] n^\mu_{\Sigma_0},$$

which is a version of *Noether's Theorem.* There are other situations in which it is desirable to have bulk terms with a sign (see for example Proposition 3.6.4).

The following proposition and its corollary give the essential definiteness properties that make the energy currents compatible with Sobolev estimates.

Proposition 2.2.2. *Let V and W be two future directed timelike vector fields. Then for any function ψ, $\mathbb{T}[\psi](V,W)$ is positive definite. By continuity, if V or W is null then $\mathbb{T}[\psi](V,W)$ is non-negative definite.*

Proof. The proof is an immediate application of the Cauchy–Schwarz inequality. □

The redshift estimate

A subextremal Kerr–Newman spacetime possesses the following Killing field

$$K := T + \frac{a}{(r_+^2+a^2)^2}\Phi = T + \frac{a}{Mr_+ - Q^2}\Phi \tag{2.2.4}$$

known as the *Hawking vector field.* It is a null generator of $\mathcal{H}^+$. Therefore, the event horizon is a *Killing horizon.* Note the identity

$$\nabla_K K = \kappa K \quad \text{where} \quad \kappa = \frac{r_+ - r_-}{2(r_+^2+a^2)} > 0.$$

The quantity κ is the *surface gravity.* The positivity of κ allows for the construction of a nondegenerate energy on (and near) the horizon which has the divergence properties needed to prove energy estimates [DR09] (see Theorem 2.2.3).

Remark In the extremal case, $\kappa = 0$, so this stabilising mechanism breaks down. In fact, our results do not hold in the extremal case! Aretakis has recently made great progress in classifying the stability and instabilities of extremal black holes, see for example [Are11], [Are12a] and [Are12b].

The following vector field will be useful:

$$K_e := T + \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2}\Phi = T + \frac{a(2Mr - Q^2)}{(r^2 + a^2)^2}\Phi. \tag{2.2.5}$$

This vector field has the following important properties.

Lemma 2.2.2. *The vector field K_e defined by* (2.2.5) *is null on the horizon $\mathcal{H}^+$ and timelike in $\mathcal{M} \setminus \mathcal{H}^+$.*

Proof. We compute $g(K_e, K_e)$. First, note that $\Delta = (r - r_+)(r - r_-)$ so $\Delta = 0$ on the horizon. Therefore, the vector field defined by (2.2.5) reduces to (2.2.4) on the horizon. A simple computation shows that $g(K_e, K_e)|_{r=r_+} = g(K, K)|_{r=r_+} = 0$.

Off the horizon, we need to show that

$$\begin{aligned}&\rho^2 g\left(T + \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2}\Phi, T + \frac{a(r^2 + a^2 - \Delta)}{(r^2 + a^2)^2}\Phi\right)\\ &= -\Delta + \sin^2\theta\left(a^2 - \frac{a^2(r^2 + a^2 - \Delta)^2}{(r^2 + a^2)^2} - \frac{\Delta a^4(r^2 + a^2 - \Delta)^2\sin^2\theta}{(r^2 + a^2)^4}\right) < 0.\end{aligned}$$

We need only consider the case that the term in parentheses is positive. It then suffices to show that

$$-\Delta + a^2 - \frac{a^2(r^2 + a^2 - \Delta)^2}{(r^2 + a^2)^2} < 0.$$

Multiplying through by $-(r^2 + a^2)^2$, we would like to have

$$(\Delta - a^2)(r^2 + a^2)^2 + a^2(r^2 + a^2 - \Delta)^2 > 0.$$

Now

$$\begin{aligned}(\Delta - a^2)(r^2 + a^2)^2 + a^2(r^2 + a^2 - \Delta)^2 &= \Delta(r^2 + a^2)^2 + \Delta^2(r^2 + a^2)^2 - 2a^2\Delta(r^2 + a^2)\\ &= (\Delta(r^2 + a^2)[r^2 - a^2 + \Delta(r^2 + a^2)] > 0,\end{aligned}$$

since $r > M > a$. $\square$

Lemma 2.2.3. *There exists an ϵ_0 such that K (defined by* (2.2.4)*) is timelike for $r \in (r_+, r_+ + \epsilon_0)$.*

Proof. We consider $r_e = r_+ + \epsilon$ for $\epsilon < \epsilon_0$ and note that $\Delta > 0$ for $r > r_+$ and $\Delta = 0$ at $r = r_+$, so $\Delta(r_e) = O(\epsilon)$. So

$$\begin{aligned}
&\rho^2 g\left(T + \frac{a}{(r_+^2+a^2)}\Phi, T + \frac{a}{(r_+^2+a^2)}\Phi\right)\\
&= -\Delta + \sin^2\theta\left(a^2 - \frac{2a^2(r^2+a^2)}{(r_+^2+a^2)} - \frac{a^2(r^2+a^2)}{(r_+^2+a^2)^2}\right) + O(\epsilon)\\
&= -\Delta + \sin^2\theta\left(\frac{a^2[(r_+^2-r^2)-a^2-r^2]}{(r_+^2+a^2)} - \frac{a^2(r^2+a^2)}{(r_+^2+a^2)^2}\right) + O(\epsilon) < 0,
\end{aligned}$$

taking ϵ_0 small enough and noting that $r > r_+$. □

Dafermos and Rodnianski showed in [DR13] that for all stationary black hole spacetimes with Killing horizons of positive surface gravity, there exists a timelike vector field N whose multiplier current $\mathbb{J}^N_\mu[\psi]$ captures the red-shift effect. In the Kerr–Newman case, we have

Theorem 2.2.3. *Let $a^2 + Q^2 < M^2$, $g = g_{M,a,Q}$ be a Kerr–Newman metric and $\mathfrak{D}^+(\Sigma_0)$ be as defined in §2.2.1. There exist positive constants $b = b(a,Q,M)$ and $B = B(a,Q,M)$, parameters $r_1(a,Q,M) > r_e(a,Q,M) > r_+$ and a φ_t-invariant timelike vector field $N = N(a,Q,M)$ on $\mathfrak{D}$ such that*

1. $\mathbb{K}^N[\psi] \geq b\,\mathbb{J}^N_\mu[\psi]N^\mu$ *for* $r \leq r_e$,
2. $-\mathbb{K}^N[\psi] \leq B\,\mathbb{J}^N_\mu[\psi]N^\mu$ *for* $r \geq r_e$,
3. $N = T$ *for* $r \geq r_1$,

where the currents are defined with respect to g.

One of the most important uses of this current is that it yields the following estimate for nondegenerate energy near the horizon, *provided we can control the last term on the right hand side of* (2.2.6).

Proposition 2.2.4. *[DR10a] Let $a^2 + Q^2 < M^2$ and $\Sigma_\tau = \varphi_\tau(\Sigma_0)$. For all $r_+ < r_0 \leq r_e$ and $\delta > 0$, there exists a constant $B_0 = B(\Sigma_0, r_0, \delta)$ such that for all solutions of* (2.2.3) *we have*

$$\begin{aligned}
&\int_{\mathcal{R}(0,\tau)\cap\{r_+\leq r\leq r_0\}} \left[\mathbb{J}^N[\psi]\cdot N + (|r-r_+|^{-1}\log|r-r_+|^{-2})\psi^2\right]\\
&+ \int_{\mathcal{H}^+(0,\tau)} \mathbb{J}^N[\psi]\cdot n_{\mathcal{H}^+} + \int_{\Sigma_\tau\cap\{r_+\leq r\leq r_0\}} \mathbb{J}^N[\psi]\cdot n_{\Sigma_\tau}\\
\leq\; & B_0\int_{\Sigma_0} \mathbb{J}^N[\psi]n^\mu_{\Sigma_0} + B_0\int_{\mathcal{R}(0,\tau)\cap\{R_0\leq r\leq r_0+\delta\}} \left[\mathbb{J}^N[\psi]\cdot n_{\Sigma_0} + \psi^2\right]. \qquad (2.2.6)
\end{aligned}$$

In what follows, we sometimes use the shorthand $|\partial\psi|^2$ to denote the quantity controlled by $\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_\tau}$, where n_{Σ_τ} is the normal to $\Sigma_\tau = \{t^* = \tau\}$. More explicitly, let us set

$$|\partial\psi|^2 = (\partial_{t^*}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2. \tag{2.2.7}$$

In what follows, we often need to estimate a weighted L^2 norm of a function by energy quantities (which contain only derivatives). This is accomplished by using Hardy inequalities of the following form:

For a square-integrable, differentiable function f vanishing as $x \to \infty$ with square integrable derivative:

$$\int_0^\infty f^2(x)dx \leq C\int_0^\infty x^2\left(\frac{df}{dx}\right)^2 dx. \tag{2.2.8}$$

To prove a Hardy inequality, one integrates the right hand sign by parts, eliminates the boundary terms and identifies a factor on the right hand side as the square root of the left hand side. The bound then follows by an application of the L^2 Cauchy–Schwarz inequality.

Chapter 3

Stability of subextremal Kerr–Newman spacetimes for linear scalar perturbations

3.1 Introduction

Our primary goal here is the proof of the following energy estimates:

- nondegenerate energy bounds

$$\int_{\Sigma_\tau} E[\psi] \leq C \int_{\Sigma_0} E[\psi], \tag{NEB}$$

- integrated local energy decay

$$\int_0^\tau \int_{\Sigma_s} E_\natural[\psi] + r^{-\alpha}\psi^2 ds \leq C \int_{\Sigma_0} E[\psi], \tag{ILED}$$

where $\alpha > 3$ and $E[\psi]$ and $E_\natural[\psi]$ are appropriately weighted square sums of the first derivatives of ψ and Σ_0 and Σ_τ are spacelike hypersurfaces. See Theorem 3.2.1 for the precise statement of these results. Higher order and pointwise decay results are then derived from these key estimates.

The energy $E_\natural[\psi]$ in (ILED) is necessarily degenerate due to the presence of trapped null geodesics. However, trapping is an obstacle to decay but not boundedness. Therefore, (NEB) should not 'see' trapping. That is, $E[\psi]$ in (NEB) is nondegenerate and both sides of the inequality (NEB) contain only first order derivatives of ψ. In this chapter, we extract the fully *nondegenerate* energy bound (NEB) from a *degenerate* integrated local energy statement (ILED) *with no loss of differentiability.* To achieve this, the precise degeneration of (ILED) due to trapping must be understood in phase space.

In this thesis we resolve the linear stability problem for scalar perturbations for the Kerr–Newman family of spacetimes by adapting the strategy of [DR11a] and [DRSR14].

3.1.1 Key elements

The following properties of subextremal Kerr spacetimes are crucial to the proof of (NEB) and (ILED) given in [DR11a] and [DRSR14]:

1. The span of the Killing vector fields T and Φ is timelike away from the horizon.
2. The wave equation can be separated as discovered by Carter in [Car68]. This separation can be carried out rigorously by restricting attention to solutions of the wave equation that are sufficiently integrable in time.
3. Superradiance and trapping are frequency specific phenomena and can be captured in disjoint regions of phase space. This means they can be dealt with separately, circumventing the need to deal with their possibly nontrivial interaction.

4. The separated wave equation possesses an algebraic structure that allows for mode stability results to be proved for solutions of the wave equation supported only on a compact range of frequencies. This was first discovered by Whiting in the celebrated [Whi89]. Whiting's result has recently been extended and quantified in [SR13]. It is the quantitative estimates of [SR13] which are necessary in the argument of [DRSR14].

5. For solutions of the wave equation supported only on a single fixed azimuthal frequency, superradiance and trapping are disjoint in physical space.

6. The metric depends smoothly on the parameter a. This allows for the restriction to sufficiently integrable solutions of the wave equation by a continuity argument in that parameter.

Miraculously, all the properties listed above have analogues in the Kerr–Newman case. Property (4) is proved in Chapter 4. The other properties of the subextremal Kerr–Newman spacetimes are proved here and used to prove that these spacetimes are stable with respect to linear scalar perturbations, i.e. (NEB) and (ILED).

3.1.2 Overview

The main results of this chapter are the precise versions of (NEB) and (ILED) for the subextremal Kerr–Newman spacetimes, stated in Theorem 3.2.1.

The proof of Theorem 3.2.1 begins in §3.3 with the restriction of attention to solutions ψ of the wave equation that are sufficiently integrable. Under this assumption, we prove (ILED) by first Fourier transforming ψ. This in turn allows for the use of Carter's separation of the wave equation.

This separation reduces the problem to analysing a second order linear ODE with potential V. The behaviour of this potential captures superradiance and trapping. The analysis of this potential in §3.4 leads to the miraculous confinement of superradiance and trapping in disjoint regions of phase space. The significance of this fact is that it allows us to construct bespoke energy multipliers to prove the phase space analogue of (ILED) in each frequency regime, see §3.5. Furthermore, this phase space version of (ILED) provides the precise understanding of trapping required to prove (NEB).

There is one further obstruction in that the frequency localised energy current used to deal with superradiance exploits a large frequency parameter. In the low frequency superradiant regime, no frequency localised energy current is available, making it difficult to obtain quantitative estimates directly. In the Kerr case, the mode stability result of [Whi89] has been recently refined in [SR13]. This refinement allows for *quantitative* estimates for the bounded superradiant frequencies in the Kerr case. The analogous mode stability results and quantitative estimates for the Kerr–Newman case are proved in Chapter 4. The low frequency estimates are presented here in §3.5.6.

The (ILED) statement then follows from summing up and inverse Fourier transforming the estimates of §3.5 and the quantitative mode stability result of Chapter 4. This is presented in §3.6.

We then remove the integrability assumption by a continuity argument in the parameter Q in §3.7. This is analogous to the continuity argument in a for the Kerr case in [DRSR14]. The proof of the (ILED) estimates under the integrability assumption is essentially the closedness part of the argument. Non-emptiness follows from the analogous (ILED) and (NEB) results for the Kerr case.

To prove openness we first note in §3.7.1 that it suffices to work with modes of fixed azimuthal frequency. We then observe that for modes of fixed azimuthal frequency, trapping is disjoint from superradiance in physical space (see Lemma 3.4.5). That is, the degeneracy in the (ILED) statement is supported strictly outside the ergoregion. We take advantage of this fact in §3.7.3 to prove a derivative-gaining (ILED)-type estimate. Finally, we define an interpolating metric and use the derivative-gaining estimate to prove that the set of subextremal Kerr–Newman spacetimes for which solutions of the wave equation are sufficiently integrable is indeed open.

In §3.8, the (NEB) statement is extracted from the (ILED) statement by constructing bespoke physical space energy currents for solutions of the wave equation localised around fixed degeneracies of the (ILED) estimate.

3.2 Main results

Our main result is a quantitative energy bound and energy decay result for solutions of (2.2.3) on the full range of subextremal Kerr–Newman spacetimes.

3.2.1 The main results: (NEB) and (ILED)

Theorem 3.2.1. *Let $a^2 + Q^2 \leq K_0^2 < M^2$. Let $g = g_{M,a,Q}$ be a subextremal Kerr–Newman metric and Σ_0 be the Cauchy hypersurface described in 2.2.1. Let ψ be a solution of* (2.2.3)*. For any $\delta > 0$ and any $r_+ < R_e < \infty$, there exist constants $C_{R_e} = C_{R_e}(K_0, M)$ and $C_\delta = C_\delta(K_0, M)$, such that we have the following: For $\tau \geq 0$ (including the limit $\tau \to \infty$), the following estimates hold:*

- *Nondegenerate energy bound*

$$\int_{\Sigma_\tau} \mathbb{J}^N[\psi] \cdot n_{\Sigma_\tau} \leq C \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0} \tag{NEB}$$

- *Integrated local energy decay for arbitrary $r_+ < R_e < \infty$*

$$\int_0^\tau \int_{\Sigma_\tau \cap \{r_+ \le r \le R_e\}} \left(\chi_\natural \mathbb{J}^N[\psi] \cdot n_{\Sigma_\tau} + |\psi - \psi_\infty|^2 \right) dt^* + \int_{\mathcal{H}^+} \mathbb{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+} \le C_{R_e} \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0} \qquad \text{(ILED)}$$

- *Integrated local energy decay up to null infinity*

$$\begin{aligned} &\int_0^\tau \int_{\Sigma_\tau} \left(r^{-1} \chi_\natural |\nabla\!\!\!/ \psi|^2 + r^{-1-\delta} \chi_\natural (T\psi)^2 + r^{-1-\delta} (Z\psi)^2 + r^{-3-\delta} |\psi - \psi_\infty|^2 \right) dt^* \\ &\quad + \int_{\mathcal{H}^+} \mathbb{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{I}^+} \mathbb{J}^T_\mu[\psi] n^\mu_{\mathcal{I}^+} \\ &\le C_\delta \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0}, \end{aligned} \tag{3.2.1}$$

  where $4\pi\psi_\infty = \lim_{R\to\infty} \int_{\Sigma_0 \cap \{r=R\}} r^{-2}\psi^2$.

Here $\chi_\natural$ is a cut-off function that vanishes in a neighbourhood of the physical space projection of the trapped set, see (3.6.2)*. In fact, a stronger version of* (ILED) *is proved in phase space, namely* (3.5.11)*. The constants C and C_{R_e} blow up as $K_0 \to M$.*

We first prove the estimates stated in the theorem under the assumption that ψ is sufficiently integrable to allow Fourier transform, see §3.3.1 and Theorem 3.3.2. The integrability assumption allows us to rigorously apply Carter's separation to the wave equation (2.1.8) in §3.3.3 and derive the phase space estimate (3.5.11) in §3.5. This follows the approach taken by Dafermos–Rodnianski for the Kerr case in [DR11a] and Dafermos–Rodnianski–Shlapentokh-Rothman in [DRSR14].

In proving (3.5.11), we come across a low-frequency obstruction which is overcome by appealing to the quantitative mode stability result Theorem 4.5.1, which itself is a generalisation of the analogous Kerr result of [SR13].

The physical space result (ILED) is retrieved from the phase space result in §3.6.

We then remove the integrability assumption by a continuity argument in the parameter Q in §3.7. This is analogous to the continuity argument in a for the Kerr case in [DRSR14]. The continuity argument makes use of the fortuitous fact that trapping occurs outside the ergoregion for solutions of (2.1.8) supported on fixed azimuthal frequencies (see Lemma 3.4.5).

Finally, we extract (NEB) from (3.5.11) in §3.8. Extensive use is made of the precise understanding of trapping in phase space and the feature of the Kerr–Newman geometry that the span of T and Φ is timelike off the horizon.

The following higher order statement is useful for various applications, e.g., proving pointwise estimates.

Theorem 3.2.2. *Under the hypotheses of Theorem 3.2.1, for all $\delta > 0$ and all integers $j \geq 1$, there exists a constant $C = C(j, \delta, M)$ such that for all $\tau \geq 0$ (including the limit $\tau \to \infty$), the following inequalities hold:*

$$\int_0^\tau \int_{\Sigma_s} r^{-1-\delta} \left(\sum_{1 \leq i_1+i_2+i_3 \leq j-1} \left(\left| \nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3+1} \psi \right|^2 + \left| \nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi \right|^2 \right) \right.$$
$$\left. + \chi_\natural \sum_{1 \leq i_1+i_2+i_3 \leq j} \left| \nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi \right|^2 \right) ds$$
$$+ \int_{\mathcal{H}^+(0,\tau)} \sum_{1 \leq i \leq j-1} \mathbb{J}^N_\mu [N^i \psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{I}^+} \sum_{1 \leq i \leq j-1} \mathbb{J}^N_\mu [N^i \psi]_{\mathcal{I}^+}$$
$$\leq C \int_{\Sigma_0} \sum_{0 \leq i \leq j-1} \mathbb{J}^N [N^i \psi] \cdot n_{\Sigma_0}, \tag{3.2.2}$$

$$\int_{\Sigma_\tau} \sum_{0 \leq i \leq j-1} \mathbb{J}^N [N^i \psi] \cdot n_{\Sigma_\tau} \leq C \int_{\Sigma_0} \sum_{0 \leq i \leq j-1} \mathbb{J}^N [N^i \psi] \cdot n_{\Sigma_0}. \tag{3.2.3}$$

Proof. Once Theorem 3.2.1 has been established, this higher order result follows by commuting the wave equation with the vector fields T, Φ and Y and applying elliptic estimates, see [DRSR14, §10] for the details. □

3.2.2 Decay results

Once (NEB) and (ILED) have been proved, one can invoke the Dafermos–Rodnianski method of [DR10b] to obtain more explicit quantitative decay results. The Dafermos–Rodnianski method makes use of a different foliation to that described in §2.2.1. Let ς_0 be a spacelike hypersurface terminating on $\mathcal{H}^+$ and asymptoting to null infinity (rather than i^0). The explicit form of such hypersurfaces is not important in the analysis, though examples may be found in [DR10b] and [DR10a]). A detailed treatment of the Dafermos–Rodnianski method allowing for a large class of spacetimes (including Kerr–Newman) and very general asymptotics may be found in [Mos].

The foliation is then defined as before: let φ_τ denote the 1-parameter family of diffeomorphisms generated by the vector field T, and define the hypersurfaces

$$\varsigma_\tau = \varphi_\tau(\varsigma_0).$$

Corollary 3.2.3. *With the foliation described above and under the hypotheses of Theorem 3.2.1, we have, in addition to* (NEB) *and* (ILED),

- *Explicit integrated energy decay*

$$\int_{\tau}^{2\tau}\int_{\varsigma_\tau\cap\{r_+\leq r\leq R_e\}} |\partial\psi|^2 d\tau \leq C_{R_e} D\tau^{-2} \tag{3.2.4}$$

- *Explicit decay of energy flux*

$$\int_{\varsigma_\tau} f(r)(\partial_{t^*}\psi)^2 + (\partial_r\psi)^2 + |\nabla\!\!\!/\psi|^2 \leq CD\tau^{-2}, \tag{3.2.5}$$

where $f(r)$ is a positive, bounded function, such that $f(r)\to 0$ as $r\to\infty$. The explicit form of f depends on the choice of ς_0 (see [SR13, Lemma D.4] for more details). The constant D denotes the square of a weighted higher-order Sobolev norm of the initial data and τ is the time function of the foliation $\cup_{\tau\geq 0}\varsigma_\tau$.

The Dafermos–Rodnianski method is a "black box" result. It requires (NEB) and the nondegenerate form of (ILED) (obtained by commuting with the Killing fields) as input and outputs (3.2.4) and (3.2.5).

One may obtain the following higher order boundedness and decay results through commutation arguments (for the details of such arguments, see [Sch12] and [DRSR14, §10])

Corollary 3.2.4. *With the foliation by hypersurfaces $\cup_{\tau\geq 0}\varsigma_\tau$ and under the hypotheses of Theorem 3.2.1, for any $\delta>0$ there exists a constant $C=C(K_0,M,\delta,R_e)>0$ such that*

$$\begin{aligned} \sup_{\varsigma_\tau} r|\psi-\psi_\infty| &\leq C\sqrt{D}\tau^{-\frac{1}{2}} \\ \sup_{\varsigma_\tau\cap\{r\leq R_e\}} |\psi-\psi_\infty| &\leq C\sqrt{D}\tau^{-3/2+\delta} \\ \text{and} \quad \sup_{\varsigma_\tau\cap\{r\leq R_e\}} (|n_{\varsigma_\tau}\psi|+|\nabla_{\varsigma_\tau}\psi|) &\leq C\sqrt{D}\tau^{-2+\delta} \end{aligned}$$

Here D denotes the square of a weighted higher-order Sobolev norm of the initial data and τ is the time function of the foliation $\cup_{\tau\geq 0}\varsigma_\tau$.

3.3 Frequency localisation

3.3.1 Assumptions

Before we begin the proof, we discuss an integrability assumption that allows us to perform phase space analysis and a criterion that ensures we have good asymptotics near the horizon and spacelike infinity.

Integrability assumption

Carter's separation of the wave equation requires Fourier transform in the t variable but, a priori, solutions of the wave equation (2.2.3) could grow exponentially in time. We therefore restrict attention to a class of functions which are assumed to be sufficiently integrable in the following sense:

Definition 3.3.1. *Let $a^2 + Q^2 \leq K_0^2 < M$ and let $g = g_{a,Q,M}$. A smooth function $\Psi : \mathcal{M} \to \mathbb{R}$ is said to be sufficiently integrable if for every $j \geq 1$ and $R > r_+$, we have*

$$\sup_{r\in[r_+,R]} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \sum_{0\leq i_1+i_2+i_3\leq j} \left|\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \Psi\right|^2 + \left|\nabla\!\!\!/^{i_1} T^{i_2} (Z^*)^{i_3} \Box_g \Psi\right|^2 \sin\theta \, dt \, d\theta \, d\phi < \infty. \tag{3.3.1}$$

Under this assumption, Ψ and its derivatives may be Fourier transformed.

The outgoing condition

We introduce a condition that will imply that solutions of the radial ODE (3.3.12) have outgoing boundary conditions.

Definition 3.3.2. *Let $K_0 < M$ and $a^2 + Q^2 \leq K_0^2$. A smooth function Ψ is said to be outgoing if there exists an $\epsilon > 0$ such that for all $\tau \leq -\epsilon^{-1}$,*

$$\begin{cases} \Psi = 0 & \text{in } \Sigma_\tau \cap \{r \leq r_+ + \epsilon\}, \\ \Psi = 0 & \Sigma_\tau \cap \{r \geq \epsilon^{-1}\} \\ \text{and } \Box_{g_{M,a,Q}} \Psi = 0 & \text{for sufficiently large } r. \end{cases} \tag{3.3.2}$$

The outgoing condition ensures that Ψ is supported away from the past event horizon and away from past null infinity. Heuristically, this means that there is no energy entering $\mathcal{M}$. We instantiate the class of outgoing, sufficiently integrable functions by applying an appropriate cut-off $\psi_{✂} = \gamma\psi$ for ψ satisfying (2.2.3).

Cutting off in time

Let ψ be a solution of (2.2.3). Define

$$\psi_{✂} = \gamma\psi, \tag{3.3.3}$$

where γ is a smooth cut-off function such that

$$\gamma(t^*) = \begin{cases} 0 & \text{for } t^* \leq 0 \\ 1 & \text{for } t^* \geq 1. \end{cases}$$

This ensures that $\psi_{✂}$ will satisfy the outgoing condition (3.3.2) and upon inverse Fourier transform, we will be able to control ψ in terms of initial data on Σ_0. The cost of this is that $\psi_{✂}$ satisfies the inhomogeneous wave equation

$$\Box_g \psi_{✂} = F, \quad \text{where} \quad F = (\Box\gamma)\psi + 2\nabla^\mu \gamma \nabla_\mu \psi. \tag{3.3.4}$$

Note that F is supported in

$$S_\gamma := \{0 \leq t^* \leq 1\} \subset \bigcup_{0<t^*<\tau} \Sigma_t.$$

This is convenient for controlling various error terms that arise in §3.6 and §3.8 by using the following

Proposition 3.3.1. *Given any $\tau \geq 0$, we have the following bound on the growth of the energy of a solution ψ of* (2.2.3)*,*

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau} \leq e^{P\tau} \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0} \tag{3.3.5}$$

for some uniform positive constant P.

Proof. Apply the energy identity for N and use (2.2.2), Theorem 2.2.3 and (3.3.4) to control the spacetime integral of $\mathbb{K}^N[\psi]$ by the spacetime integral of $\mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}$ and apply Grönwall's inequality. $\square$

By the reduction argument in [DR10a, §4.6], we can, without loss of generality, assume further that ψ that arises from smooth, compactly supported data on Σ_0. It follows that ψ is compactly supported on all Σ_τ for $\tau \geq 0$ and $\lim_{x \to i^0} r|\psi(x)|^2 = 0$.

3.3.2 The conditional version of (ILED)

From here until the end of §3.6 we work with solutions ψ of (2.2.3) that arise from smooth, compactly supported data on Σ_0 and associated $\psi_{✂}$ that are assumed to satisfy (3.3.2) and (3.3.1). Under these assumptions we prove the following conditional theorem.

Theorem 3.3.2. *Let $a^2 + Q^2 \leq K_0^2 < M^2$. Let $g = g_{M,a,Q}$ be a subextremal Kerr–Newman metric and Σ_0 be the Cauchy hypersurface described in 2.2.1. Let ψ be a solution of* (2.2.3) *arising from smooth, compactly supported data on Σ_0. Assume that $\psi_{✂}$ defined by* (3.3.3) *satisfies* (3.3.1) *and* (3.3.2)*. For any $\delta > 0$ and any $r_+ < R_e < \infty$, there exist constants $C_{R_e} = C_{R_e}(K_0, M)$ and $C_\delta = C_\delta(K_0, M)$ such that the following estimates hold for all $\tau \geq 0$ (including the limit $\tau \to \infty$):*

- *Integrated local energy decay for arbitrary* $r_+ < R_e < \infty$

$$\int_0^\infty \int_{\Sigma_\tau \cap \{r_+ \leq r \leq R_e\}} \left(\chi_\natural |\partial\psi|^2 + |\psi - \psi_\infty|^2\right) dt^* \leq C_{R_e} \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0}. \tag{3.3.6}$$

- *Integrated local energy decay up to null infinity*

$$\begin{aligned} &\int_0^\tau \int_{\Sigma_\tau} \left(r^{-1}\chi_\natural |\nabla\!\!\!/\psi|^2 + r^{-1-\delta}\chi_\natural (T\psi)^2 + r^{-1-\delta}(Z\psi)^2 + r^{-3-\delta}|\psi - \psi_\infty|^2\right) dt^* \\ &\qquad + \int_{\mathcal{H}^+} \mathbb{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+} + \int_{\mathcal{I}^+} \mathbb{J}^T_\mu[\psi] n^\mu_{\mathcal{I}^+} \\ &\leq C_\delta \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0}, \end{aligned} \tag{3.3.7}$$

Here $\chi_\natural$ *is a cut-off function that vanishes in a neighbourhood of the physical space projection of the trapped set, see* (3.6.2)*. In fact, a stronger version of* (ILED) *is proved in phase space, namely* (3.5.11)*.*

As stated, Theorem 3.3.2 may be read as the improvement of a soft nonquantitative statement to a uniform quantitative statement. In the case of axisymmetry, the assumption that ψ is sufficiently integrable function can easily be removed in light of [DR10a] and [DR11b]. In the general case, it can be seen in the context of a continuity argument. From this point of view, Theorem 3.3.2 corresponds to the closedness part (see §3.7.4) and the removal of the restriction (3.3.1) is then openness (see §3.7.3).

3.3.3 Carter's separation

Carter discovered in [Car68] that the wave equation on a Kerr–Newman background can be formally separated.

In the Kerr case, the authors of [DR10a], [DR11a] and [DRSR14] used Carter's separation as a geometric framework to derive frequency localised energy estimates. In particular, the frequency localisation captures trapping and superradiance in disjoint regions of phase space. This is the key observation in proving (ILED). This new approach highlights salient properties of individual modes and unites classical mode analysis and the vector field method, both of which have long histories in the literature. Here we generalise this approach to the Kerr–Newman spacetimes.

Oblate spheroidal harmonics

The separation of the wave equation will require the decomposition of ψ into *oblate spheroidal harmonics.* Let $\xi \in \mathbb{R}$ and consider the following elliptic operator acting on the dense subset of $L^2(\mathbb{S}^2)$ formed by the smooth functions on $\mathbb{S}^2$:

$$P_\xi f = -\not{\Delta}_{\mathbb{S}^2} f - (\xi^2 \cos^2\theta) f.$$

We gather some useful facts from elliptic PDE theory in the following proposition.

Proposition 3.3.3. *Let $\xi \in \mathbb{R}$. The eigenvalues $\lambda_{m\ell}^{(\xi)}$ of P_ξ are real with corresponding eigenfunctions of the form $S_{m\ell}^{(\xi)}(\cos\theta)e^{im\phi}$. These eigenfunctions constitute a complete orthonormal basis for $L^2(\mathbb{S}^2)$ and satisfy:*

$$\left(P_\xi - \lambda_{m\ell}^{(\xi)}\right) S_{m\ell}^{(\xi)}(\cos\theta)e^{im\phi} = 0 \quad \text{with } m, \ell \in \mathbb{Z},\ \ell \geq |m|.$$

The functions $S_{m\ell}^{(\xi)}(\cos\theta)$ are smooth in ξ and θ and $\lambda_{m\ell}^{(\xi)}$ are smooth in ξ. Further

$$\begin{aligned} \lambda_{m\ell}^{(\xi)} + \xi^2 &\geq |m|(|m|+1) \\ \text{and} \quad \lambda_{m\ell}^{(\xi)} + \xi^2 &\geq 2|m\xi|. \end{aligned}$$

For $\xi = 0$, these simplify to the standard spherical harmonics $S_{m\ell}^{(0)} = Y_{m\ell}$ and $\lambda_{m\ell}^{(0)} = \ell(\ell+1)$.

Proof. See [Are12a, §8.2]. □

The functions $S_{m\ell}^{(\xi)}$ are known as *oblate spheroidal harmonics.* Turning to the Kerr–Newman geometry, we will apply Proposition 3.3.3 with $\xi = a\omega$, where ω is the phase space variable associated to Fourier transforming $\psi_{\text{✂}}$ in time.

Performing the separation

Let ψ be and $\psi_{\text{✂}}$ be as in Theorem 3.3.2. Since $\psi_{\text{✂}}$ satisfies the integrability assumption (3.3.1), it may be Fourier transformed in t. This allows us to separate the wave equation (2.1.8) by first Fourier transforming $\Box_g \psi_{\text{✂}} = F$ in t and expanding in terms of the oblate spheroidal harmonics. This leads to the following decomposition:

$$\psi_{\text{✂}}(t,r,\theta,\phi) = \overbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \underbrace{\sum_{m,\ell>|m|} R_{m\ell}^{(a\omega)}(r) \cdot S_{m\ell}^{(a\omega)}(\cos\theta) e^{im\phi}}_{\text{Oblate spheroidal expansion}} e^{-i\omega t}}^{\text{Fourier expansion}} \, d\omega \tag{3.3.8}$$

where $S^{(a\omega)}_{m\ell}(\cos\theta)$ and $\lambda^{(a\omega)}_{m\ell}$ are the eigenfunctions and eigenvalues of $P_{a\omega}$ respectively and

$$R^{(a\omega)}_{m\ell}(r) = \int_{\mathbb{S}^2} \widehat{\psi_{\text{✂}}}(\omega, r, \theta, m) \cdot \overline{S^{(a\omega)}_{m\ell}(\cos\theta) e^{im\phi}} dV_{\mathbb{S}^2}. \tag{3.3.9}$$

Moreover, $R^{(a\omega)}_{m\ell}(r)$ satisfies the radial equation

$$\left[\partial_r(\Delta\partial_r) - \omega^2\left(a^2 - \frac{(a^2+r^2)^2}{\Delta}\right) + \frac{a^2m^2}{\Delta} - \frac{2am\omega(2Mr - Q^2)}{\Delta} - \lambda^{(a\omega)}_{m\ell}\right] R^{(a\omega)}_{m\ell} = F^{(a\omega)}_{m\ell} \tag{3.3.10}$$

in the sense of $L^2(d\omega)\ell^2(m,\ell)$. It is convenient to work with the coordinate r^*. Note that in this coordinate system

$$\{t, r^*, \theta, \phi\} = \mathbb{R}\times\mathbb{R}\times[0,\pi]\times[0,2\pi),$$

whereas Boyer–Lindquist coordinate patch was only valid (modulo degeneration of angular coordinates) in the range $r \in (r_+, \infty)$.

We now define

$$u^{(a\omega)}_{m\ell}(r^*) = \sqrt{r^2+a^2} R^{(a\omega)}_{m\ell}(r). \tag{3.3.11}$$

Writing (3.3.10) in the new coordinates, we have the *radial Carter ODE*:

$$\frac{d^2}{(dr^*)^2} u^{(a\omega)}_{m\ell}(r^*) + \left(\omega^2 - V^{(a\omega)}_{m\ell}(r)\right) u^{(a\omega)}_{m\ell} = H^{(a\omega)}_{m\ell}, \tag{3.3.12}$$

where equality is meant in the sense of $L^2(d\omega)\ell^2(m,\ell)$ and

$$\begin{aligned}
V^{(a\omega)}_{m\ell}(r) &= \frac{2am\omega(2Mr-Q^2) - a^2m^2 + \Delta\cdot\Lambda^{(a\omega)}_{m\ell}}{(r^2+a^2)^2} \\
&\quad + \frac{\Delta(3r^2+a^2+Q^2-4Mr)}{(r^2+a^2)^3} - \frac{3\Delta^2r^2}{(r^2+a^2)^4}, \\
H^{(a\omega)}_{m\ell}(r) &= \frac{\Delta F^{(a\omega)}_{m\ell}(r)}{(r^2+a^2)^{1/2}} \\
\text{and } \Lambda^{(a\omega)}_{m\ell} &= \lambda^{(a\omega)}_{m\ell} + a^2\omega^2, \\
\text{which obeys } \Lambda^{(a\omega)}_{m\ell} &\geq |m|(|m|+1) \qquad (3.3.13)\\
\text{and } \Lambda^{(a\omega)}_{m\ell} &\geq 2|am\omega|. \qquad (3.3.14)
\end{aligned}$$

Note that even though $R^{(a\omega)}_{m\ell}$ is complex-valued, the potential $V^{(a\omega)}_{m\ell}$ is real.

Physical space–Fourier space identities

The following identities are immediate consequences of Parseval's identity and Plancherel's identity. They form the bridge between frequency-localised estimates and physical space estimates.

For any fixed $r > r_+$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \sum_{m,\ell} \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega &= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (\psi_{\text{✂}})^2 \cdot (r^2+a^2)\; dt\; dg_{\mathbb{S}^2}, \\
\int_{-\infty}^{+\infty} \sum_{m,\ell} \omega^2 \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega &= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (T\psi_{\text{✂}})^2 \cdot (r^2+a^2)\; dt\; dg_{\mathbb{S}^2}, \\
\int_{-\infty}^{+\infty} \sum_{m,\ell} \left| u_{m\ell}^{(a\omega)\prime} \right|^2 d\omega &= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} \left(\partial_{r^*} \left(\sqrt{r^2+a^2} \cdot \psi_{\text{✂}} \right) \right)^2 \; dt\; dg_{\mathbb{S}^2},
\end{aligned}$$

and

$$\begin{aligned}
\int_{-\infty}^{+\infty} \sum_{m,\ell} \Lambda_{m\ell}^{(a\omega)} \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega &= \int_{-\infty}^{+\infty} \sum_{m,\ell} (\lambda_{m\ell}^{(a\omega)} + a^2\omega^2) \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega \\
&= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (-\slashed{\Delta}_{\mathbb{S}^2} - a^2\omega^2\cos^2\theta + a^2\omega^2)|u|^2 d\omega\; dg_{\mathbb{S}^2} \\
&= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (|\slashed{\nabla}_{\mathbb{S}^2} u|^2 + a^2\omega^2 \sin^2\theta |u|^2) d\omega\; dg_{\mathbb{S}^2} \\
&= \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (|\slashed{\nabla}_{\mathbb{S}^2} \psi_{\text{✂}}|^2 + a^2 \sin^2\theta (T\psi_{\text{✂}})^2) \cdot (r^2+a^2)\; dt\; dg_{\mathbb{S}^2},
\end{aligned}$$

The identities above immediately imply that for any fixed $r > r_+$:

$$\begin{aligned}
\int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (\psi_{\text{✂}})^2 \cdot r^2\; dt\; dg_{\mathbb{S}^2} &\leq \int_{-\infty}^{+\infty} \sum_{m,\ell} \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega, \\
\int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (T\psi_{\text{✂}})^2 \cdot r^2\; dt\; dg_{\mathbb{S}^2} &\leq \int_{-\infty}^{+\infty} \sum_{m,\ell} \omega^2 \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega, \\
\int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} |\slashed{\nabla}\psi_{\text{✂}}|^2 r^2\; dt\; dg_{\mathbb{S}^2} &\leq \int_{-\infty}^{+\infty} \sum_{m,\ell} \Lambda_{m\ell}^{(a\omega)} \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega \\
\text{and} \quad \int_{-\infty}^{+\infty} \int_{\mathbb{S}^2} (\partial_{r^*}\psi_{\text{✂}})^2 \cdot r^2\; dt\; dg_{\mathbb{S}^2} &\leq \int_{-\infty}^{+\infty} \sum_{m,\ell} 2\left| u_{m\ell}^{(a\omega)\prime} \right|^2 + 8\left| u_{m\ell}^{(a\omega)} \right|^2 d\omega.
\end{aligned}$$

Boundary conditions

In view of the cut-off γ, the solution $\psi_{\text{✂}}$ of (3.3.4) satisfies (3.3.2).

Recall that $K = T + \frac{a}{2Mr_+ - Q^2}\Phi$ is the null generator of the future horizon $\mathcal{H}^+$. Since

$\partial_{r^*} \to K$ as $r^* \to -\infty$ and $\psi_{✂}$ is smooth at $r = r_+$, Plancherel implies the following asymptotic condition on $u^{(a\omega)}_{m\ell}(r^*)$ near the horizon:

$$\int_{-\infty}^{\infty} \sum_{m,\ell} \left| (u^{(a\omega)}_{m\ell})'(r^*) + i\left(\omega - \frac{am}{2Mr_+ - Q^2}\right) u^{(a\omega)}_{m\ell}(r^*) \right|^2 d\omega \tag{3.3.15}$$

is smooth in r and tends to 0 as $r \to r_+$.

A similar argument shows that (3.3.2) implies the following asymptotic condition on $u^{(a\omega)}_{m\ell}(r^*)$ for large r (see [DRSR14, §5.3-5.4] for the details): There exists a sequence $\{r_n\}_{n=1}^{\infty}$ such that $r_n \to \infty$ and

$$\lim_{n\to\infty} \left| (u^{(a\omega)}_{m\ell})'(r_n) - i\omega u^{(a\omega)}_{m\ell}(r_n) \right| = 0 \quad \text{for almost every } \omega. \tag{3.3.16}$$

Here, and in all that follows, u' denotes a derivative with respect to r^*.

Almost everywhere regularity

The analysis that follows is focused on (3.3.12), which holds for $u^{(a\omega)}_{m\ell}(r^*) \in L^2(d\omega)\ell^2(m,\ell)$. It is much more convenient to consider smooth solutions $u^{(a\omega)}_{m\ell}$ of (3.3.12) satisfying the boundary conditions (3.3.15) and (3.3.16).

Definition 3.3.3. *Let* $\psi_{✂}$ *satisfy* (3.3.1) *and* (3.3.2) *and define* $u^{(a\omega)}_{m\ell}(r^*)$ *by* (3.3.11) *and* (3.3.9). *Define* $\Omega \subset \mathbb{R}$ *to be the set of frequencies* ω *such that for all* m *and* ℓ, $H^{(a\omega)}_{m\ell}$ *is smooth and* $u^{(a\omega)}_{m\ell}$ *is a smooth solution of* (3.3.12) *satisfying the boundary conditions* (3.3.15) *and* (3.3.16).

The following lemma allows for the reduction to classical solutions.

Lemma 3.3.4. *The set* $\{\omega \in \mathbb{R}\} \setminus \Omega$ *has measure zero.*

Proof. See [DRSR14, Lemma 5.4.1]. □

Since we will integrate over $\omega \in \mathbb{R}$ to prove the physical space (ILED) estimate, it suffices to prove frequency-localised energy estimates hold for almost every ω. Therefore, we may restrict attention to smooth solutions $u^{(a\omega)}_{m\ell}$ of (3.3.12) by considering $\omega \in \Omega$.

3.3.4 Frequency-localised energy current templates

We now turn our attention to the task of generating frequency-localised estimates. That is, we will prove energy estimates for each $u^{(a\omega)}_{m\ell}$ with $\omega \in \Omega$. To do this, we need frequency-localised analogues of the energy currents of §2.2.3. In this section, templates for such currents are described. Note that the dependence of u, H and V on $a\omega, m$ and ℓ is suppressed in this section. Let us also reiterate the warning that we use the notation $f' := \frac{df}{dr^*}$.

The frequency-localised conserved energy currents

The frequency-localised analogue of the conserved energy current $\mathbb{J}^T_\mu[\psi]$ is

$$Q_T[u] = \omega \mathrm{Im}[u'\bar{u}]. \tag{3.3.17}$$

From this we compute

$$\begin{aligned} Q'_T[u] &= \omega \mathrm{Im}[u''\bar{u} + \left|u'\right|^2] \\ &= \omega \mathrm{Im}[H\bar{u} - (\omega^2 - V)|u|^2] \\ &= \omega \mathrm{Im}[H\bar{u}]. \end{aligned}$$

The conservation identity for the Q_T current is

$$\int_{r_+}^{\infty} Q'_T[u](r)\ dr = Q_T[u](\infty) - Q_T[u](r_+). \tag{3.3.18}$$

The boundary conditions (3.3.15) and (3.3.16) imply that

$$Q_T[u](r_+) = -\omega\left(\omega - \frac{am}{2Mr_+ - Q^2}\right)|u(r_+)|^2 \quad \text{and} \quad Q_T[u](\infty) = \omega^2|u(\infty)|^2, \tag{3.3.19}$$

where

$$\omega_+ := \frac{am}{2Mr_+ - Q^2}. \tag{3.3.20}$$

Clearly, $Q_T[u](\infty) \geq 0$. Denoting

$$\mathcal{G}_{☼} := \{(\omega, m)\ :\ \omega(\omega - \omega_+) < 0\}, \tag{3.3.21}$$

we have non-negativity of $-Q_T[u](r_+)$ for $(\omega, m) \notin \mathcal{G}_{☼}$. We refer to $\mathcal{G}_{☼}$ as the *superradiant regime*. Thus the bulk term on the left hand side of (3.3.18) is positive in the non-superradiant regime.

Since we assume $a \geq 0$, (3.3.21) is equivalent to

$$\mathcal{G}_{☼} = \{(\omega, m)\ :\ m\omega \in (0, m\omega_+)\}. \tag{3.3.22}$$

We will use this simpler condition when discussing superradiance. Recall that the assumption $a \geq 0$ is made with no loss of generality, see 2.2.2.

The frequency-localised analogue of the conserved energy current $\mathbb{J}^K_\mu[\psi]$ is

$$Q_K[u] = (\omega - \omega_+)\mathrm{Im}[u'\bar{u}]. \tag{3.3.23}$$

From this we compute

$$Q_K'[u] \quad = \quad (\omega - \omega_+)\mathrm{Im}[H\bar{u}].$$

The boundary conditions (3.3.15) and (3.3.16) imply that

$$Q_K[u](r_+) = -\left(\omega - \frac{am}{2Mr_+ - Q^2}\right)^2 |u(r_+)|^2 \quad \text{and} \quad Q_K[u](\infty) = \omega(\omega - \omega_+)|u(\infty)|^2. \tag{3.3.24}$$

The frequency-localised virial currents

The frequency-localised analogues of virial currents $\mathbb{J}^{X,w}_\mu[\psi]$ (where X is in the ∂_{r^*} direction and w is some function) are naturally constructed from combinations of the following templates: for arbitrary piecewise differentiable $f(r^*)$, $h(r^*)$ and $y(r^*)$, define

$$Q_0^f[u] = f[\left|u'\right|^2 + (\omega^2 - V)|u|^2] + f'\mathrm{Re}(u'\bar{u}) - \frac{1}{2}f''|u|^2, \tag{3.3.25}$$

$$Q_1^h[u] = h\mathrm{Re}(u'\bar{u}) - \frac{1}{2}h'|u|^2, \tag{3.3.26}$$

$$Q_2^y[u] = y[\left|u'\right|^2 + (\omega^2 - V)|u|^2]. \tag{3.3.27}$$

Note that $Q_0^f = Q_1^{f'} + Q_2^f$. We compute

$$\begin{aligned}
(Q_0^f)' &= 2f'\left|u'\right|^2 - fV'|u|^2 - \frac{1}{2}f'''|u|^2 + 2f\mathrm{Re}(u'\bar{H}) + f'\mathrm{Re}(u\bar{H}), \\
(Q_1^h)' &= h[\left|u'\right|^2 + (V - \omega^2)|u|^2] - \frac{1}{2}h''|u|^2 + h\mathrm{Re}(u\bar{H}), \\
(Q_2^y)' &= y'[\left|u'\right|^2 + (\omega^2 - V)|u|^2] - yV'|u|^2 + 2y\mathrm{Re}(u'\bar{H}),
\end{aligned}$$

where we have made repeated use of the Carter ODE (3.3.12) and the simple identity

$$2\mathrm{Re}(w\bar{z}) = w\bar{z} + \bar{w}z.$$

The frequency-localised red-shift current

The frequency-localised analogue of the red-shift current $\mathbb{J}^N_\mu$ is

$$Q^z_{red} = z\left[\left|u' + i\left(\omega - \omega_+\right)u\right|^2 + \left(\omega^2 - V - |\omega - \omega_+|^2\right)|u|^2\right]. \tag{3.3.28}$$

Recall that $\Delta = 0$ at $r = r_+$. This allows us to characterise ω_+ by

$$\omega^2 - V(r_+) = |\omega - \omega_+|^2. \tag{3.3.29}$$

Coupling this to the boundary conditions (3.3.15) and (3.3.16) we have

$$\begin{aligned} \left|u'(r_+)\right|^2 &= (\omega - \omega_+)^2|u(r_+)|^2 && (3.3.30)\\ \text{and} \quad \left|u'(\infty)\right|^2 &= \omega^2|u(\infty)|^2. && (3.3.31)\end{aligned}$$

In §3.5.2, the function z is chosen in such a way that $z \to \infty$ as $r \to r_+$ to produce a finite, non-zero boundary term for Q^z_{red}.

Let $\tilde{V} = V + |\omega - \omega_+|^2 - \omega^2$. Then $\tilde{V}(r_+) = 0$ and $\tilde{V}'(r) = V'(r)$. Note that we are referring to the value of r^* for which $r = r_+$. We compute

$$(Q^z_{red})' = z'\left|u' + i\left(\omega - \omega_+\right)u\right|^2 - (z\tilde{V})'|u|^2 + 2z\mathrm{Re}\left((u' + i(\omega - \omega_+)u)\bar{H}\right).$$

3.4 Properties of the potential

In §3.5, we will use the templates of §3.3.4 to derive frequency localised energy estimates analogous to (ILED). The proof of these estimates hinges on the properties of the potential V in the Carter ODE (3.3.12).

This argument in the Kerr case is due to Dafermos and Rodnianski and was first given in the survey paper [DR11a]. The analysis of the potential V is extended to the Kerr–Newman case here. In particular, the miraculous fact that trapping and superradiance occur in disjoint regions of phase space in the Kerr case carries over to the Kerr–Newman case. Furthermore, for solutions of (2.2.3) supported only on a single azimuthal frequency, superradiance and trapping are disjoint in physical space as well. These properties of the subextremal Kerr–Newman spacetimes are extremely fortunate as there is no a priori reason to expect them to follow from the Kerr case.

Consider the potential $V(r)$ for $r \geq r_+$. We begin by decomposing V into its frequency dependent and independent parts:

$$
\begin{aligned}
V &= V_0 + V_1, \\
\text{where}\quad V_0 &= \frac{2am\omega(2Mr - Q^2) - a^2m^2 + \Delta\Lambda}{(r^2+a^2)^2} && (3.4.1)\\
\text{and}\quad V_1 &= \frac{\Delta}{(r^2+a^2)^4}((3r^2 - 4Mr + a^2 + Q^2)(r^2+a^2) - 3\Delta r^2) \\
&= \frac{\Delta}{(r^2+a^2)^4}(a^2\Delta + 2Mr^3 - 2Q^2r^2) \\
&> \frac{\Delta}{(r^2+a^2)^4}(a^2\Delta + 2r^2(M^2 - Q^2)). && (3.4.2)
\end{aligned}
$$

We immediately see that $V_1 \geq 0$. We will focus mainly on proving properties of V_0 and show that these properties carry over (with slight perturbation) to V.

3.4.1 Critical points of the potential

Lemma 3.4.1. *For $\omega \in \Omega$ and any m, Λ:*

1. *The potential function V_0 has at most one maximum, r^0_{max}, and one minimum, r^0_{min}, on the interval (r_+, ∞).*
2. *If these extrema are achieved, $r^0_{min} < r^0_{max}$.*
3. *For all sufficiently large Λ, the value r^0_{max} is bounded uniformly from above provided that either $m\omega \geq 0$ or $a^2\omega^2 \leq C\Lambda$ for some constant C. In the latter case the bound for r^0_{max} may depend on C.*

Proof. Compute

$$
\begin{aligned}
\frac{d}{dr}V_0 &= \frac{4am\omega M}{(r^2+a^2)^2} + \frac{(4r)\left(a^2m^2 - 2am\omega(2Mr - Q^2)\right)}{(r^2+a^2)^3} \\
&\quad + \frac{\Lambda}{(r^2+a^2)^3}\left[2(r-M)(r^2+a^2) - 4\Delta r\right].
\end{aligned}
$$

So

$$
\begin{aligned}
(r^2+a^2)^3\frac{d}{dr}V_0 &= -12am\omega Mr^2 + 4a^2m^2r + 4am\omega[Q^2r + Ma^2] \\
&\quad -2\Lambda\left[r^3 - 3Mr^2 + (2Q^2 + a^2)r + a^2M\right]. && (3.4.3)
\end{aligned}
$$

Compute

$$\begin{aligned}\frac{d}{dr}\left[(r^2+a^2)^3\frac{d}{dr}V_0\right] &= -24am\omega Mr + 4a^2m^2 + 4am\omega Q^2\\ &\quad -2\Lambda\left[3r^2 - 6Mr + (2Q^2+a^2)\right].\end{aligned}$$

Letting $\sigma = \dfrac{am\omega}{\Lambda}$,

$$\begin{aligned}\frac{d}{dr}\left[(r^2+a^2)^3\frac{d}{dr}V_0\right] = -6\Lambda\Bigg[r^2 - 2Mr + 4Mr\sigma - \frac{2}{3}\left(Q^2\sigma + \frac{a^2m^2}{\Lambda}\right)\\ +\frac{1}{3}(a^2+2Q^2)\Bigg].\end{aligned}$$

The zeros of this function lie at

$$r_{1,2} = M(1-2\sigma) \pm \sqrt{M^2(1-2\sigma)^2 - \frac{1}{3}\left[a^2(1-\frac{2m^2}{\Lambda}) + 2Q^2(1-\sigma)\right]}.$$

We now consider the cases $\sigma \geq 0$ and $\sigma < 0$ separately.

Case $\sigma \geq 0$: In this case $r_2 < M < r_+$ so the only point where $\frac{d}{dr}\left[(r^2+a^2)^3\frac{d}{dr}V_0\right]$ can vanish on (r_+,∞) is

$$r_1 = M(1-2\sigma) + \sqrt{M^2(1-2\sigma)^2 - \frac{1}{3}\left[a^2(1-\frac{2m^2}{\Lambda}) + 2Q^2(1-\sigma)\right]}.$$

Observe that $\Lambda > 0$ by Proposition 3.3.3, so (3.4.3) implies that

$$(r^2+a^2)^3\frac{d}{dr}V_0 \to -\infty \quad \text{as} \quad r\to\infty.$$

Since r_1 is the only root of $\frac{d}{dr}\left[(r^2+a^2)^3\frac{d}{dr}V_0\right]$ and $(r^2+a^2)^3$ and its derivative are positive on (r_+,∞), $\frac{d}{dr}V_0$ has at most two zeros r^0_{min} and r^0_{max}, which must be the extrema corresponding to their subscripts since

$$\frac{d^2}{dr^2}V_0(r^0_{max}) < 0 \quad \text{and} \quad \frac{d^2}{dr^2}V_0(r^0_{min}) > 0.$$

Further, $r^0_{min} < r^0_{max}$.

If it turns out that $r_1 \notin \mathbb{R}$, then $\frac{d}{dr}\left[(r^2+a^2)^3\frac{d}{dr}V_0\right]$ is negative for all $r \geq r_+$, so $\frac{d}{dr}V_0$ can vanish at only one point, where V_0 must attain a maximum.

Case $\sigma < 0$: In this case we need to do some work to show that $r_2 < M$. We factorise

$$r_2 = M(1-2\sigma)\left[1 - \sqrt{1 - \frac{a^2(1-\frac{2m^2}{\Lambda}) + 2Q^2(1-\sigma)}{3M^2(1-2\sigma)^2}}\right].$$

Since $\sigma < 0$, $a^2 + Q^2 < M^2$ and $\Lambda > m^2$:

$$\left|\frac{a^2(1-\frac{2m^2}{\Lambda})}{3M^2(1-2\sigma)^2}\right| < \frac{1}{3} \quad \text{and} \quad \left|\frac{2Q^2(1-\sigma)}{3M^2(1-2\sigma)^2}\right| < \frac{2}{3}.$$

Also, for $0 \leq x < 1$, $\sqrt{1-x} > 1-x$. Thus $1 - \sqrt{1-x} < x$. This implies

$$\begin{aligned} r_2 &< M(1-2\sigma)\left[\frac{a^2(1-\frac{2m^2}{\Lambda}) + 2Q^2(1-\sigma)}{3M^2(1-2\sigma)^2}\right] \\ &= \left[\frac{a^2(1-\frac{2m^2}{\Lambda}) + 2Q^2(1-\sigma)}{3M(1-2\sigma)}\right] \\ &< \frac{a^2+2Q^2}{3M} < \frac{3M^2}{M}. \end{aligned}$$

So $r_2 < M$ and we may argue as in the previous case.

For the last statement of the lemma, we observe that as $r \to \infty$, the behaviour of $(r^2+a^2)^3 \frac{d}{dr} V_0$ is governed by $(6\Lambda - 12am\omega M)r^2 - 2\Lambda r^3$. For $(6\Lambda - 12am\omega M)r^2 - 2\Lambda r^3 = 0$,

$$r = 3M\left(1 - \frac{6am\omega}{\Lambda}\right).$$

So if $m\omega \geq 0$ or $a^2\omega^2 \leq C\Lambda$ this quantity is bounded, consequently bounding r^0_{max}. □

3.4.2 Superradiant frequencies are not trapped

Lemma 3.4.2. *For $\omega \in \Omega$ and any m, Λ:*

$$V(r_+) \leq \omega^2$$

with equality only for $\omega = \omega_+$. In particular, this implies

$$V_0(r_{min}) \leq \omega^2.$$

Proof. Recall that $\Delta = r_+^2 - 2Mr_+ + a^2 + Q^2 = 0$ on the horizon and compute

$$\begin{aligned}
\omega^2 - V(r_+) &= \omega^2 - \frac{2am\omega(2Mr_+ - Q^2) - a^2m^2}{(r_+^2 + a^2)^2} \\
&= \omega^2 - \frac{2am\omega(2Mr_+ - Q^2) - a^2m^2}{(2Mr_+ - Q^2)^2} \\
&= \frac{\omega^2(2Mr_+ - Q^2)^2 - 2am\omega(2Mr_+ - Q^2) + a^2m^2}{(2Mr_+ - Q^2)^2} \\
&= \frac{\big(\omega(2Mr_+ - Q^2) - am\big)^2}{(2Mr_+ - Q^2)^2},
\end{aligned}$$

which is manifestly non-negative. □

This result means that if r_{min} exists, it can only be 'trapped' for the threshold value of the superradiant regime:

$$\omega = \omega_+ = \frac{am}{2Mr_+ - Q^2}.$$

By Lemma 3.4.1, For r_{min} to exist, it is necessary that $\frac{d}{dr}V(r_+) < 0$. The next lemma shows that this is not the case for superradiant frequencies. Therefore, in the superradiant regime, V_0 has only a maximum (r_{min} is absent).

Lemma 3.4.3. *Let $\omega \in \Omega$. If $m\omega \leq \dfrac{am^2}{2Mr_+ - Q^2}$, then there exists a $c > 0$ such that*

$$\frac{d}{dr}V(r_+) \geq \frac{d}{dr}V_0(r_+) \geq c\Lambda > 0.$$

Proof. To show the first inequality it suffices that

$$\frac{d}{dr}V_1(r_+) = \frac{2r_+^2(r_+ - M)(Mr_+ - Q^2)}{(r_+^2 + a^2)} > 0. \tag{3.4.4}$$

We now show that $\frac{d}{dr}V_0(r_+)$ is positive. Noting that $r_+^2 + a^2 = 2Mr_+ - Q^2$, we have

$$\begin{aligned}
(r_+^2 + a^2)^3 \frac{d}{dr}V_0(r_+) &= (4am\omega M)(2Mr_+ - Q^2) + 4a^2m^2r_+ - 8amr\omega(2Mr_+ - Q^2) \\
&\quad + 2\Lambda(r_+ - M)(2Mr_+ - Q^2).
\end{aligned} \tag{3.4.5}$$

Using the superradiant condition we have

$$\begin{aligned}
(2Mr_+ - Q^2)^2 \frac{d}{dr} V_0(r_+) &= 4am\omega M + 4am\omega r_+ \\
&\quad -8am\omega r_+ + 2\Lambda(r_+ - M) \\
&= 4am\omega(M - r_+) + 2\Lambda(r_+ - M) \\
&= 2(r_+ - M)(\Lambda - 2am\omega) \\
&\geq 2(r_+ - M)\left(\Lambda - \frac{2a^2m^2}{2Mr_+ - Q^2}\right) \\
&\geq 2(r_+ - M)\left(\frac{(2Mr_+ - Q^2)\Lambda - 2a^2m^2}{2Mr_+ - Q^2}\right) \\
&> 2m^2(r_+ - M)\left(\frac{2Mr_+ - Q^2 - 2a^2}{2Mr_+ - Q^2}\right)
\end{aligned}$$

where we have used Proposition 3.3.3. Positivity holds since $r_+ > M$. □

Corollary 3.4.1. *The conclusion of Lemma 3.4.3 can be extended to the range*

$$am\omega \leq \frac{a^2m^2}{2Mr_+ - Q^2} + \alpha\Lambda, \quad \omega \in \Omega, \tag{3.4.6}$$

for sufficiently small constant α.

Proof. A small negative term just appears in our estimates:

$$\begin{aligned}
(r_+^2 + a^2)^2 \frac{d}{dr} V_0(r_+) &= 4am\omega M + 4am\omega r_+ - 4\alpha\Lambda r_+ - 8am\omega r_+ + 2\Lambda(r_+ - M) \\
&= 4am\omega(M - r_+) + 2\Lambda(r_+ - M) - 4\alpha\Lambda r_+ \\
&= 2(r_+ - M)(\Lambda - 2am\omega) - 4\alpha\Lambda r_+.
\end{aligned}$$

Using (3.4.6) we have

$$\begin{aligned}
\frac{d}{dr} V_0(r_+) &\geq \frac{2(r_+ - M)}{(r_+^2 + a^2)^2}\left(\Lambda - 4\alpha\Lambda r_+ - \frac{2a^2m^2}{2Mr_+ - Q^2}\right) - \frac{(4\alpha\Lambda r_+)}{(r_+^2 + a^2)} \\
&\geq \frac{2(r_+ - M)}{(r_+^2 + a^2)^2}\left(\frac{(2Mr_+ - Q^2)(1 - 4\alpha r_+)\Lambda - 2a^2m^2}{2Mr_+ - Q^2}\right) - \frac{(4\alpha\Lambda r_+)}{(r_+^2 + a^2)} \\
&> 2m^2(r_+ - M)\left(\frac{(2Mr_+ - Q^2)(1 - 4\alpha r_+) - 2a^2}{2Mr_+ - Q^2}\right) - \frac{(4\alpha\Lambda r_+)}{(r_+^2 + a^2)},
\end{aligned}$$

so choosing α small enough we retain positivity. □

The next result mathematically embodies the miraculous disunion of the superradiant and trapped frequencies.

Lemma 3.4.4. *For all $a^2 + Q^2 < M^2$, $\omega \in \Omega$ and $0 \leq m\omega \leq m\omega_+ + \alpha\Lambda$ there exists a $k > 1$ such that*

$$\omega^2 - V(r_{max}) < \omega^2 - V_0(r^0_{max}) < \frac{\Delta}{2(r_+^2 + a^2)^2}\left[m^2 - k\Lambda\right] < 0. \tag{3.4.7}$$

Proof. It suffices to prove the lemma with $\alpha = 0$.

We first consider the case when $m\left(\frac{am}{2Mr_+ - Q^2} - \omega\right) \leq \epsilon\,|m|\,\sqrt{\Lambda}$. In this case we have

$$\omega^2 - V_0(r_+) = \left(\omega - \frac{am}{2Mr_+ - Q^2}\right)^2 \leq \epsilon^2\Lambda.$$

Combining this with Corollary 3.4.1, we have

$$V_0(r_+ + \delta) - \omega^2 \geq b\Lambda$$

for some sufficiently small $\delta > 0$ and even smaller ϵ.

In the case where $\omega^2 \leq \epsilon\Lambda$, we have

$$V_0(r) - \omega^2 \geq \frac{\Lambda}{r^2} + O\left(\frac{\Lambda}{r^3}\right) - \epsilon\Lambda \text{ as } r \to \infty.$$

So taking $\tilde{r}$ sufficiently large and letting ϵ be sufficiently small,

$$V_0(\tilde{r}) - \omega^2 \geq b\Lambda.$$

Finally, consider the case where $m\left(\frac{am}{2Mr_+} - \omega\right) > \epsilon\,|m|\,\sqrt{\Lambda}$ and $\omega^2 > \epsilon\Lambda$.

Pick r_0 such that

$$m\omega = \frac{am^2}{2Mr_0 - Q^2}.$$

Then $\omega(2Mr_0 - Q^2) = am$.

In this case, r_0 will satisfy $r_0 \in [r_+ + \delta, R]$ for some $\delta > 0$ and $R < \infty$.

Now compute

$$\begin{aligned}
\omega^2 - V_0(r_0) &= \frac{\left[(r_0^2+a^2)^2\omega^2 - 2am\omega(2Mr_0 - Q^2) + a^2m^2 - \Delta\Lambda\right]}{(r_0^2+a^2)^2} \\
&= \frac{\left[(r_0^2+a^2)^2\omega^2 - \omega^2(2Mr_0 - Q^2)^2 - \Delta\Lambda\right]}{(r_0^2+a^2)^2} \\
&= \frac{\left[\omega^2[(r_0^2+a^2)^2 - (2Mr_0 - Q^2)^2] - \Delta\Lambda\right]}{(r_0^2+a^2)^2} \\
&= \frac{\Delta}{(r_0^2+a^2)^2}\left[\omega^2(r_0^2+a^2+2Mr_0-Q^2) - \Lambda\right] \\
&= \frac{\Delta}{(r_0^2+a^2)^2}\left[\frac{a^2m^2(r_0^2+a^2+2Mr_0-Q^2)}{(2Mr_0-Q^2)^2} - \Lambda\right] \\
&< \frac{\Delta}{(r_0^2+a^2)^2}\left[\frac{a^2m^2r_0^2}{(2Mr_0-Q^2)^2}(1+\frac{a^2}{r_0^2}+\frac{2M}{r_0}) - \Lambda\right].
\end{aligned}$$

But

$$\begin{aligned}
(2Mr_0 - Q^2)^2 &= 4M^2r_0^2 - 4Q^2Mr_0 + Q^4 \\
&> 4M^2r_0^2 - 4Q^2r_0^2.
\end{aligned}$$

Since $a^2 < r_+ - \delta < r_0$,

$$\begin{aligned}
\omega^2 - V_0(r_0) &< \frac{\Delta}{(r_0^2+a^2)^2}\left[\frac{a^2m^2}{4(M^2-Q^2)}(1+\frac{a^2}{r_0^2}+\frac{2M}{r_0}) - \Lambda\right] \\
&< \frac{\Delta}{(r_0^2+a^2)^2}\left[m^2(1-\delta) - \Lambda\right]
\end{aligned}$$

which is negative by Proposition 3.3.3.

It is immediate that

$$\omega^2 < V_0(r^0_{max}) \leq V(r^0_{max}) \leq V(r_{max})$$

so that the characterisation above of the disunion of the superradiant and trapped frequencies holds for the full potential. □

3.4.3 Trapping for fixed azimuthal mode solutions

The following lemma shows that if we fix the azimuthal frequency m, then trapping occurs outside the ergoregion.

Lemma 3.4.5. *Let λ_2 be a potentially small parameter and let λ_1 and ω_1 be potentially large parameters, all of which are to be determined in §3.5. Recall that $\sigma = \frac{am\omega}{\Lambda}$. Let m*

be fixed, $\omega \in \Omega$ and let (ω, Λ) lie in the trapped frequency regime

$$\mathcal{F}^m_\natural = \left\{(\omega, \Lambda) \ : \ |\omega| > \omega_1, \quad \lambda_2 \Lambda \leq \omega^2 \leq \lambda_2^{-1}\Lambda\right\}.$$

There exists a constant $c > 0$ such that the conditions $|\sigma| \leq c$, $m^2 \leq c\Lambda$, $c^{-1} \leq \Lambda$ and $a^2 + Q^2 \leq K_0^2 < M^2$ imply that $r^0_{max} > (1+\sqrt{2})M$.

Proof. We showed in the proof of Lemma 3.4.1 that

$$\begin{aligned}(r^2+a^2)^3 \frac{d}{dr} V_0 &= -12am\omega M r^2 + 4a^2m^2r + 4am\omega[Q^2r + Ma^2] \\ &\quad -2\Lambda\left[r^3 - 3Mr^2 + (2Q^2+a^2)r + a^2M\right]\end{aligned}$$

By the same lemma, r^0_{max} is the largest critical point of V_0, so it suffices to show that $\frac{d}{dr}V_0(r = (1+\sqrt{2})M) > 0$.

We compute

$$\begin{aligned}\Lambda^{-1} \quad & (r^2+a^2)^3 \frac{d}{dr} V_0|_{(r=(1+\sqrt{2})M)} \\ = \quad & -12\sigma M^2(1+\sqrt{2})^2 + 4\frac{a^2m^2}{\Lambda}(1+\sqrt{2})M + 4\sigma M[Q^2(1+\sqrt{2}) + a^2] \\ & -2\left[(1+\sqrt{2})^3M^3 - 3M^2(1+\sqrt{2})^2 + (2Q^2+a^2)(1+\sqrt{2})M + a^2M\right] \\ = \quad & 4\sigma\left[-3M^2(1+\sqrt{2})^2 + MQ^2(1+\sqrt{2}) + Ma^2\right] + 4\frac{a^2m^2}{\Lambda}(1+\sqrt{2})M \\ & -2\left[(7+5\sqrt{2} - 3(3+2\sqrt{2}))M^3 + (Q^2+a^2)(2+\sqrt{2})M\right] \\ = \quad & 2(2+\sqrt{2})M\left[M^2 - (Q^2+a^2)\right] + 4\frac{a^2m^2}{\Lambda}(1+\sqrt{2})M \\ & +4\sigma M\left[-3M(3+2\sqrt{2}) + Q^2(1+\sqrt{2}) + a^2\right].\end{aligned}$$

It is only the σ term which may be non-positive, so choosing c small enough completes the proof. Since m is fixed and $\omega^2 \sim \Lambda$, $\sigma \sim \frac{1}{\sqrt{\Lambda}}$. Hence the choice of c can be made by ensuring that Λ is large enough. □

Remark The ergoregion is confined to $\left\{r < 2M < (1+\sqrt{2})M\right\}$. Thus the lemma above implies that trapping occurs outside the ergoregion for modes of fixed azimuthal frequency. This will be important in the continuity argument, see Lemma 3.7.4.

3.5 Frequency localised estimates

In this section we will construct frequency localised energy estimates that, upon summation and inverse Fourier transform, will yield the required physical space energy estimate (ILED). To do this, it is useful to exploit the frequency specific behaviour of the potential V obtained in §3.4. We begin by partitioning phase space into disjoint regimes in which the potential displays certain distinctive properties. In particular, we deal with superradiance and trapping separately in phase space.

3.5.1 Partitioning the frequency ranges

Let λ_2 be a potentially small parameter and let λ_1 and ω_1 be potentially large parameters, all of which are to be determined but are subject to the constraint

$$\lambda_2\lambda_1 = \omega_1^2. \tag{3.5.1}$$

This constraint will be enforced by choosing λ_1, and ω_1 as large as required and λ_2 as small as required, then either enlarging λ_1 or shrinking λ_2 (by a finite amount) to satisfy (3.5.1), see §3.5.7.

We decompose phase space parametrised by the frequencies ω, m and Λ as follows :

Unbounded frequencies

$$\mathcal{F}_\sharp = \{(\omega, m, \Lambda) \ : \ |\omega| > \omega_1 \ \text{ or } \ \Lambda > \lambda_1\}$$

- High superradiant frequencies

$$\mathcal{F}_{\text{☼}} = \left\{(\omega, m, \Lambda) \in \mathcal{F}_\sharp \ : \ \Lambda \geq \left(\frac{a}{2Mr_+ - Q^2} + \alpha\right)^{-2} \omega_1^2, \quad m\omega \in [0, m\omega_+ + \alpha\Lambda]\right\}$$

- Trapped frequencies

$$\mathcal{F}_\natural = \big\{(\omega, m, \Lambda) \in \mathcal{F}_\sharp \ : \ |\omega| > \omega_1, \quad \lambda_2\Lambda \leq \omega^2 \leq \lambda_2^{-1}\Lambda, \quad m\omega \notin [0, m\omega_+ + \alpha\Lambda]\big\}$$

- Time dominated frequencies

$$\mathcal{F}_{\odot} = \big\{(\omega, m, \Lambda) \in \mathcal{F}_\sharp \ : \ |\omega| > \omega_1, \quad \Lambda < \lambda_2\omega^2, \quad m\omega \notin [0, m\omega_+ + \alpha\Lambda]\big\}$$

- Angular dominated frequencies

$$\mathcal{F}_{\measuredangle} = \big\{(\omega, m, \Lambda) \in \mathcal{F}_\sharp \ : \ \Lambda > \lambda_2^{-1}\omega_1^2, \quad \omega^2 < \lambda_2\Lambda, \quad m\omega \notin [0, m\omega_+ + \alpha\Lambda]\big\}$$

Bounded frequencies

$$\mathcal{F}_{\flat,full} = \{(\omega, m, \Lambda) \ : \ |\omega| \le \omega_1, \quad 0 \le \Lambda \le \lambda_1\}$$

- Near-stationary frequencies

$$\mathcal{F}_{\flat,1} = \left\{(\omega, m, \Lambda) \in \mathcal{F}_{\flat,full} \ : \ |\omega| \le \omega_0 \ll 1, \quad 0 \le \Lambda \le \lambda_1\right\}$$

- Non-stationary frequencies

$$\mathcal{F}_{\flat,2} = \left\{(\omega, m, \Lambda) \in \mathcal{F}_{\flat,full} \ : \ \omega_0 < |\omega| \le \omega_1, \quad \Lambda \le \lambda_1\right\}$$

The following lemma shows that the partitioning above indeed covers the whole of phase space.

Lemma 3.5.1. *For all $a^2 + Q^2 < M^2$ and all (ω, m, Λ) satisfying* (3.3.13) *and* (3.3.14), *for all choices of parameters ω_1, λ_2, the triple (ω, m, Λ) lies in exactly one of the frequency ranges $\mathcal{F}_{\medstar}$, $\mathcal{F}_{\natural}$, $\mathcal{F}_{\odot}$, $\mathcal{F}_{\measuredangle}$, or $\mathcal{F}_{\flat,full}$.*

Proof. Observe that

$$|\omega| \ge \omega_1 \text{ and } m\omega \in \left(0, \frac{am^2}{2Mr_+ - Q^2} + \alpha\Lambda\right] \Rightarrow \Lambda \ge \left(\frac{a}{2Mr_+ - Q^2} + \alpha\right)^{-2} \omega_1^2.$$

Also note that the constraint (3.5.1) ensures that $\Lambda \le \lambda_2^{-1}\omega_1^2 \Rightarrow \Lambda \le \lambda_1$. This in turn implies that $\omega^2 \le \lambda_2\Lambda \Rightarrow \omega^2 < \omega_1^2$. □

As discussed in §3.3.3, we restrict attention to frequencies $\omega \in \Omega$. For these frequencies, the solutions $u_{m\ell}^{(a\omega)}$ of (3.3.12) are smooth and satisfy the boundary conditions (3.3.15) and (3.3.16).

The energy estimates for each frequency regime are presented below. The derivations of the estimates are based on those for the Kerr case. Wherever details are omitted they may be found in [DR11a, §11] for the high frequency range $\mathcal{F}_{\sharp}$ and [DRSR14, §8.7] for the low frequency range $\mathcal{F}_{\flat}$.

3.5.2 High superradiant frequencies $\mathcal{F}_{\medstar}$

This is a large frequency regime in which superradiance occurs.

Proposition 3.5.1. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\medstar}$ and suppose $0 \le a^2 + Q^2 \le K_0^2 < M^2$. Let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16).

Taking λ_1, R_∞ and E all sufficiently large, there exist functions f, h, ζ satisfying the uniform bounds

$$|f| + \Delta^{-1}r^2|f'| + |h| + |\zeta| \leq B,$$

$$f = 1,\ h = 0,\ \zeta = 0 \text{ for } r^* \geq R^*_\infty,$$

and positive frequency independent constants A and Γ such that, for sufficiently small $b > 0$,

$$\begin{aligned}
& b\left(\int_{r_+}^{R_e} (|u'|^2 + (\omega^2+\Lambda)|u|^2 dr^*) + \int_{r_+}^{r_{max}} \frac{|u' + i(\omega-\omega_+)u|^2}{r-r_+} dr^*\right) \\
& +b(\omega^2+\Lambda)\left[|u|^2|_{r=r_+} + |u|^2|_{r=\infty}\right] \\
\leq\ & -\int_{r_+}^{\infty} \left(f' + Ah\right)\mathrm{Re}(u\bar{H}) + 2f\mathrm{Re}(u'\bar{H}) - 2\Gamma\frac{\zeta}{\tilde{V}}\mathrm{Re}(u' + i(\omega-\omega_+)\bar{H})dr^* \\
& + \int_{r_+}^{\infty} E\omega\mathrm{Im}(u\bar{H})dr^*. \qquad (3.5.2)
\end{aligned}$$

The key to the proof of this estimate is that trapping does not occur in this frequency range, see Lemma 3.4.4. Before we proceed, we must take into account the frequency independent part of the potential V_1.

Lemma 3.5.2. *In the unbounded frequency regime $\mathcal{F}_\sharp$, if the potential V_0 attains a maximum at r^0_{max}, then the full potential V attains a maximum at r_{max},*

$$\left|r_{max} - r^0_{max}\right| \leq c\lambda_1^{-1}$$

Proof. Note that r^0_{max} is uniformly bounded above and away from r_+ by Lemma 3.4.1. (The lower bound follows from the fact that the maximum must lie beyond the value r_1 defined in the proof of Lemma 3.4.1). That is,

$$c \leq r^0_{max} - r_+ \leq C$$

where c and C do not depend on Λ. We first show that V has a critical point:

We know that $\frac{d}{dr}V_0 = 0$ at $r^0_{max} > r_1 + c_1$ and $(r^2+a^2)^3\frac{d}{dr}V_0 \to -\infty$ as $r \to \infty$. In light of the bound:

$$\left|\frac{d}{dr}V_1\right| \leq Cr^{-4},$$

we see that $\frac{d}{dr}V_0$ must eventually dominate $\frac{d}{dr}V_1$, so that $\frac{d}{dr}V = 0$ at some r_{max}. Furthermore, this r_{max} is also bounded above.

We now locate the maximum of V:

Since $r^0_{max} > r_1 + c_1$, we can bound

$$\frac{d}{dr}V_0 > c\lambda_1 \quad \text{in} \quad [r_{min} + \delta, r_1]$$

So for λ_1 large enough,

$$\frac{d}{dr}V_0 > c\lambda_1 - Cr^{-4} > 0 \quad \text{in} \quad [r_{min} + \delta, r_1].$$

Thus

$$\frac{d}{dr}V > c\lambda_1 > 0 \quad \text{in} \quad [r_{min} + \delta, r_1].$$

We can find $r_c \in (r_1, r^0_{max})$ such that

$$\frac{d}{dr}V_0 > \frac{c\lambda_1}{2} \quad \text{in} \quad [r_{min} + \delta, r_c]$$

and

$$\frac{d}{dr}\left((r^2 + a^2)^3 \frac{d}{dr}V_0(r)\right) \leq -c\Lambda r^2 \quad \text{in} \quad [r_c, \infty).$$

Applying the mean value theorem, there exists an r_d such that

$$\left|\frac{d}{dr}\left((r_d^2 + a^2)^3 \frac{d}{dr}V_0(r_d)\right)\right|\left|r_{max} - r^0_{max}\right| = (r_d^2 + a^2)^3 \left|\frac{d}{dr}V(r_{max}) - \frac{d}{dr}V(r^0_{max})\right|$$

Now since r_{max} and r^0_{max} are both greater than r_1, we have

$$\left|r_{max} - r^0_{max}\right| \leq \frac{c}{\lambda_1}.$$

This bound on the location of r_{max} also tells us that a maximum occurs there. □

Proof of Proposition 3.5.1. Recall from Proposition 3.3.3 that $\Lambda \geq |m|(|m|+1)$. Combining this with the superradiant condition

$$m\omega \leq \frac{am^2}{2Mr_+ - Q^2},$$

we have

$$\omega^2 \leq \left(\frac{am}{2Mr_+ - Q^2}\right)^2 < \left(\frac{a}{2Mr_+ - Q^2}\right)^2 \Lambda.$$

Also, $\omega^2 + \Lambda > \lambda_1$ in $\mathcal{F}_{☼}$, so we conclude that Λ must be very large in this regime. Thus a bound on $|u'| + \Lambda|u|^2$ will suffice.

We know from Lemmas 3.4.3 and 3.5.2, that in $\mathcal{F}_{☼}$, the potential V has only one critical point, a maximum at r_{max}, uniformly bounded above and away from r_+ by Lemma 3.4.1.

Now that we understand the behaviour of the potential in this regime we may construct our frequency-localised current from the templates (3.3.25), (3.3.26), (3.3.28) and (3.3.17):

$$Q = Q_0^f + AQ_1^h + \Gamma Q_{red}^z - EQ_T,$$

where A, E and Γ are positive frequency independent constants and f, h and z are real-valued functions depending on the frequency triple (ω, m, Λ). The purpose of each term in the above current is as follows:

- Applying Q_0^f with suitable f gives us control over a non-negative definite expression in $|u|^2 + \Lambda|u|^2$. However, the estimate we obtain will degenerate at $r = r_{max}$ due to the presence of V'. We choose f such that

$$f = \begin{cases} -1 & \text{at } \ r = r_+, \\ 0 & \text{at } \ r = r_{max}, \\ 1 & \text{for } \ r \geq R_\infty \end{cases}$$
$$\text{and} \qquad -fV' - \frac{1}{2}f''' \geq 0.$$

  We further require that $f' > 0$ and $f''' < 0$ in (r_+, R_∞). Such an f can be constructed in $[r_+, R_\infty]$ by choosing $f''' = (r - r_{max})^3$ and choosing the constants of integration appropriately. Then we have the required control

$$(Q_0^f)' \geq b(\left|u'\right|^2 + \Lambda|u|^2).$$

  in $(r_+, R_\infty) \setminus r_{max}$. (This degeneracy is due to the vanishing of f at r_{max}.)

- By (3.4.7), the degeneracy in the estimate for (Q_0^f) may be removed by adding a large multiple of the current Q_1^h, where h is a non-negative function supported in $[r_{max} - \delta, r_{max} + \delta]$ and

$$h(r) = 1 \quad \forall\, r \in (r_{max} - \delta/2, r_{max} + \delta/2).$$

- For non-superradiant frequencies, the boundary terms in the estimate for $Q_0^f + AQ_1^h$ have a favourable sign and can be controlled by simply subtracting a large multiple of Q_T. In the superradiant regime, there is a lack of control on these boundary terms as they have unfavourable sign.

  We apply the current ΓQ_{red}^z to ensure that the sum of all boundary terms on the horizon has a favourable sign. We exploit the presence of the large parameter Λ by

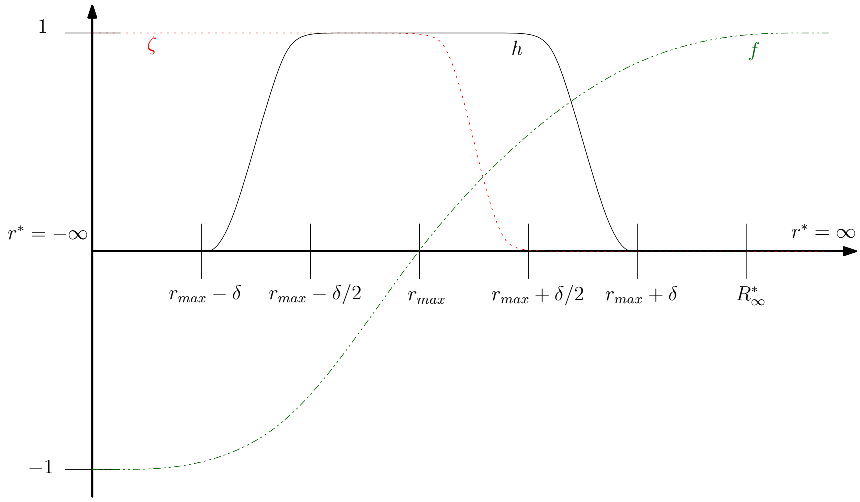

Figure 3.5.1: The functions f, h and ζ.

taking $z = -\Lambda\tilde{V}^{-1}(r)\zeta(r)$ where

$$\begin{cases} \zeta(r) = 1 & \text{for } r_+ \leq r \leq r_{max}, \\ 0 \leq \zeta(r) \leq 1 & \text{for } r_{max} < r < r_{max} + \delta/2, \\ \zeta(r) = 0 & \text{for } r_{max} + \delta/2 < r. \end{cases}$$

By choosing choosing $A \gg \Gamma$ and λ_1 large enough,

$$Ah(V - \omega^2) - Ah'' \geq A(ch\Lambda - h'') - \Gamma\Lambda \geq 0.$$

So the integrand on the left hand side of the estimate is positive and controls the necessary terms. Note that we have again made crucial use of (3.4.7).

- Since we have the large parameter Γ, we can control the boundary terms by adding the current $-EQ_T$ with $2 < E \ll \Gamma$. Then the boundary terms have the 'right' sign. This will finalise the choice of Γ and A once we have chosen E, see the remark below.

Applying the current constructed above yields the frequency-localised estimate for $\mathcal{F}_{\mathfrak{\Sigma}}$. Figure 3.5.1 illustrates the functions f, h and ζ in this construction. □

Remark Let us emphasise that f, h and ζ are real-valued functions that can be uniformly controlled so that the right hand side of (3.5.2) can be dominated by initial data (see §3.6). The constants A and Γ are frequency independent parameters, however they depend on the constant E. This constant will be used in applying the Q_T current in each frequency range. The required size of E varies in each regime but it is always a large parameter. We will finalise the choice of E in §3.5.7. This in turn finalises the choice of the constants A and Γ. The constant $b > 0$ can always be replaced with a smaller positive value without affecting the validity of (3.5.2).

3.5.3 Trapped frequencies $\mathcal{F}_\natural$

Consider the large frequency regime

$$\mathcal{F}_\natural = \left\{(\omega, m, \Lambda) \ : \ |\omega| > \omega_1, \quad \lambda_2\Lambda \leq \omega^2 \leq \lambda_2^{-1}\Lambda, \quad m\omega \notin [0, m\omega_+ + \alpha\Lambda]\right\},$$

This is the range in which trapping occurs. This means that any positive definite current controlling all derivatives must necessarily degenerate at $r = r_{max}$.

Proposition 3.5.2. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_\natural$ and suppose $0 \leq a^2 + Q^2 \leq K_0^2 < M^2$. Let u be a smooth solution of* (3.3.12) *with boundary terms* (3.3.15) *and* (3.3.16)*. Taking ω_1, R_∞ and E all sufficiently large, there exist functions f and y satisfying the uniform bounds*

$$|f| + \Delta^{-1}r^2\left|f'\right| + |y| \leq B,$$

$$f = 1, \ y = 0 \text{ for } r^* \geq R^*_\infty$$

such that, for sufficiently small $b > 0$,

$$\begin{aligned} & b\int_{r_+}^{R_e} \left[\left|u'\right|^2 + |u|^2 + (r - r_{max})^2(\omega^2 + \Lambda)|u|^2\right] dr^* \\ & +b(\omega^2 + \Lambda)\left[|u|^2|_{r=r_+} + |u|^2|_{r=\infty}\right] \\ \leq & \ -\int_{r_+}^{\infty} 2f\mathrm{Re}(u'\bar{H}) + f'\mathrm{Re}(u\bar{H}) - E\omega\mathrm{Im}(u\bar{H})dr^* - \int_{r_+}^{r_3} 2y\mathrm{Re}(u'\bar{H})dr^*. \end{aligned} \tag{3.5.3}$$

Proof. For trapped frequencies, the potential V_0 may have at most two critical points $r^0_{min} < r^0_{max}$. From Lemma 3.4.2, we know that $\omega^2 - V(r_+) \geq 0$. To construct a current, it is necessary to identify the region where $(\omega^2 - V)$ may be negative. Following the argument in [DR11a, §11.5], we find that there exists $r_3 > r_+$,

$$V(r) \leq \omega^2 - \frac{c}{4}\Lambda \quad \forall\, r \in [r_+, r_3].$$

Either r_3 is bounded above, or $r_3 = \infty$. If r_3 is finite, the potential V has a unique nondegenerate maximum at some r_{max} in $[r_3, \infty)$ which is $\Lambda^{-1}-$close to r^0_{max}.

Since $\lambda_2\Lambda \leq \omega^2 \leq \lambda_2^{-1}\Lambda$, it suffices to bound

$$\left|u'\right|^2 + \Lambda|u|^2 \ \text{ or } \ \left|u'\right|^2 + \omega^2|u|^2.$$

However, we cannot control these quantities everywhere, the best we can hope for is an estimate that degenerates precisely at r_{max}.

We use the templates (3.3.25), (3.3.26), (3.3.27) and (3.3.17) to construct a frequency-localised current as follows:

$$Q = Q_0^f - Q_2^y - EQ_T,$$

where E is a positive frequency independent constant and f and y are real-valued functions depending on the frequency triple (ω, m, Λ). Of f, we require that $f(r_+) = 0$, $f' > 0$ on $[r_3, \infty)$, $f''' < 0$ on $[r_3, R_\infty)$, f changes sign from negative to positive at $r = r_{max}$ (That is, $f(r_{max}) = 0$, $f'(r_{max}) > 0$) and $f = 1$ for $r \geq R_\infty$.

We will take y supported in $[r_+, r_3)$, with $y' < 0$ on $[r_+, r_3)$. It remains to construct y so that the left hand side of the estimate is non-negative and vanishes only at $r = r_{max}$. In summary, it suffices that y be positive, monotonically decreasing and

$$-\frac{d}{dr}y \geq -Cy + C.$$

The function

$$y = Ce^{Cr}\int_r^{r_3} e^{Cr}dr = 1 - e^{C(r-r_3)}$$

satisfies this differential inequality. For large enough E, the boundary conditions will have the right sign due to the non-superradiant condition.

Figure 3.5.2 illustrates the functions f and y in this construction.

Noting that y and f vanish precisely at $r = r_{max}$, we have the estimate for the trapping regime. $\square$

Remark The estimate for the trapped regime (3.5.3) reveals the nature of the trapped set – the estimate for each trapped frequency must degenerate at exactly one point (where V attains its maximum). The degeneration of these estimates carries over into physical space: the physical space estimate must degenerate in a neighbourhood of the physical space projection of the 'trapped set'

$$\left\{ r \ : \ r \in \bigcap_{L=1}^{\infty} \bigcup_{l \geq L} r_{m\ell}^{(a\omega)} \right\}$$

where $r_{m\ell}^{(a\omega)}$ are the points r_{max} where the potential V attains its maximum in the $\mathcal{F}_\natural$ range.

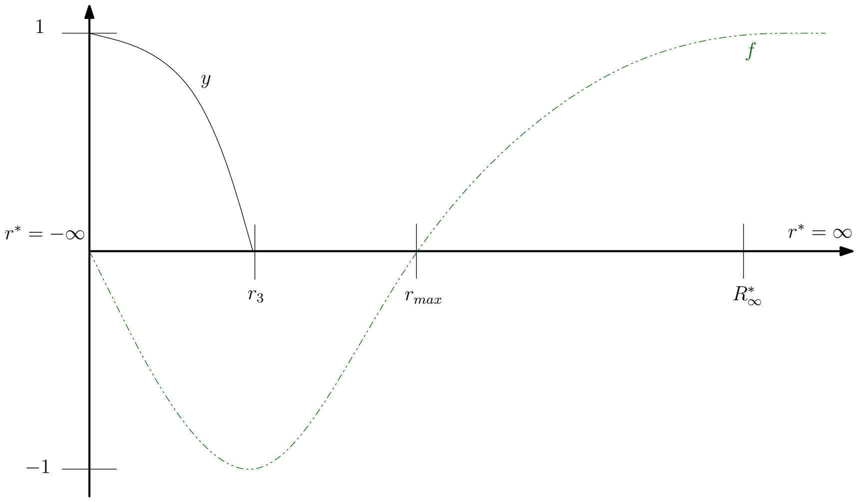

Figure 3.5.2: The functions f and y.

3.5.4 Time dominated frequencies $\mathcal{F}_{\circlearrowleft}$

We have already dealt with the superradiant and trapped frequencies, so these obstructions are not present here.

Proposition 3.5.3. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\circlearrowleft}$ and suppose $0 \leq a^2 + Q^2 \leq K_0^2 < M^2$. Let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16)*. Taking ω_1, λ_1, λ_2^{-1}, R_∞ and E all sufficiently large, there exists a function f satisfying the uniform bounds*

$$|f| \leq B \quad and \quad f = 1 \;\; for\; r^* \geq R^*_\infty,$$

such that, for sufficiently small $b > 0$,

$$\begin{aligned} & b\int_{r_+}^{R_e} \frac{\Delta}{r^5}\left[|u'|^2 + |u|^2 + (\omega^2+\Lambda)|u|^2\right] dr^* + b(\omega^2+\Lambda)[|u|^2|_{r=_+} + |u|^2|_{r=\infty}] \\ \leq \; & \int_{r_+}^{\infty} E\omega \mathrm{Im}(u\bar{H}) - 2f\mathrm{Re}(u'\bar{H}) dr^*. \end{aligned} \tag{3.5.4}$$

Proof. Here ω^2 dominates Λ so it suffices to estimate $|u'|^2 + \omega^2|u|^2$. We construct the following current from the templates (3.3.27) and (3.3.17):

$$Q = Q_2^f - EQ_T,$$

where f is monotonically increasing with $\frac{1}{2} \leq f \leq 1$ in (r_+, R_∞), $f = 1$ for $r \geq R_\infty$ and E is a positive frequency independent constant. Again, f depends on the frequency triple (ω, m, Λ). The subtraction of a large multiple of the conserved energy current EQ_T yields boundary terms with favourable sign. We obtain

$$\begin{aligned} & b\int_{r_+}^{R_e}\left[f'|u'|^2+\left(f'(\omega^2-V)-fV'\right)|u|^2\right]dr^*+b(\omega^2+\Lambda)[|u|^2|_{r=_+}+|u|^2|_{r=\infty}] \\ \leq & \int_{r_+}^{\infty} E\omega\mathrm{Im}(u\bar{H})-\mathrm{Re}(u'\bar{H})dr^* \end{aligned}$$

It just remains to check that the integrand of the left hand side of the estimate above is positive and controls the desired quantity. This is done by choosing ω_1, λ_1 and λ_2 appropriately. □

3.5.5 Angular dominated frequencies $\mathcal{F}_{\measuredangle}$

Proposition 3.5.4. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\measuredangle}$ and suppose $0 \leq a^2+Q^2 \leq K_0^2 < M^2$. Let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16)*. Taking ω_1, λ_2^{-1}, R_∞ and E all sufficiently large, there exist functions f and y satisfying the uniform bounds*

$$|f|+\Delta^{-1}r^2|f'|+|h| \leq B,$$

$$f=1,\ h=0 \text{ for } r^* \geq R^*_\infty$$

and a positive frequency independent constant A such that, for sufficiently small $b>0$,

$$\begin{aligned} & b\int_{r_+}^{R_e}\left[|u'|^2+|u|^2+(\omega^2+\Lambda)|u|^2\right]dr^*+b(\omega^2+\Lambda)[|u|^2|_{r=_+}+|u|^2|_{r=\infty}] \\ \leq & -\int_{r_+}^{\infty} 2f\mathrm{Re}(u'\bar{H})+(f'+Ah)\mathrm{Re}(u\bar{H})-E\omega\mathrm{Im}(u\bar{H})dr^*. \end{aligned} \tag{3.5.5}$$

Proof. We just repeat the construction of the current used in the proof of Proposition 3.5.1, letting

$$Q=Q_0^f+AQ_1^h-EQ_T$$

with the same f, h and A and E. The argument is simpler than in the superradiant regime as we may control the boundary terms directly. □

3.5.6 The bounded frequency range $\mathcal{F}_\flat$

This bounded low-frequency regime depends on ω_1 and λ_1 but unlike the high frequency regimes, the estimates in this section will hold for arbitrarily chosen (but finite) ω_1 and λ_1.

We will consider four subcases. This requires the introduction of a small parameter $\tilde{K}_0$ which will be chosen later in this section.

Note that we do not need to distinguish between superradiant and non-superradiant frequencies. In the near-stationary frequency range, we follow the approach of [DRSR14] with by further decomposing $\mathcal{F}_{\flat,1}$ to take advantage of small parameters, axisymmetry and non-vanishing of parameters respectively.

In light of Theorem 4.5.1, the desired estimate for the non-stationary frequencies may be obtained directly.

In view of the boundedness of the frequency parameters in $\mathcal{F}_\flat$, we have

$$\left|u'\right|^2 + (\omega^2 + \Lambda)|u|^2 \leq \max\{1, \lambda_1, \omega_1\}\left(\left|u'\right|^2 + |u|^2\right),$$

so it suffices to estimate the quantity $\left|u'\right|^2 + |u|^2$.

The near-stationary subrange (small parameters case): $|\omega| \leq \omega_0$ and $0 \leq a^2 + Q^2 \leq \tilde{K}_0^2$

Proposition 3.5.5. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\flat,1}$ and suppose $0 \leq a^2 + Q^2 \leq \tilde{K}_0^2$. Let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16)*. Then for all $\omega_1 > 0$, $\lambda_2 > 0$, sufficiently small $\omega_0 > 0$ and $\tilde{K}_0 > 0$ (depending on $\omega_1 > 0$ and $\lambda_2 > 0$), sufficiently large $R_\infty > R_e$ and $E > 2$, there exist functions y, $\hat{y}$, χ_2 and h, satisfying the uniform bounds*

$$|y| + |\hat{y}| + |h| + |\chi_2| \leq B,$$

$$y = 1,\ \hat{y} = 0,\ h = 0 \text{ for } r^* \geq R_\infty^*,$$

such that

$$\begin{aligned} & b\int_{r_e}^{R_e} \left(\left|u'\right|^2 + |u|^2\right) dr^* + b\left(\left|u'\right|^2 + \omega^2|u|^2\right)|_{r=\infty} \\ & \leq \int_{-\infty}^{\infty} \left(2(y+\hat{y})\, Re(u'\overline{H}) + h\, Re(u\overline{H}) + E\omega Im(H\overline{u}) + \chi_2\, (\omega - \omega_+)\, Im(H\overline{u})\right). \end{aligned} \tag{3.5.6}$$

Proof. We construct a current analogous to that given in [DRSR14, §8.7.1]. All that is required is that the following hold.

1. It is clear from (3.4.1) and (3.4.2) that for every $-\infty < \alpha < \beta < \infty$, we may take $\tilde{K}_0$ and ω_0 small enough that $r \in [\alpha, \beta] \Rightarrow V - \omega^2 > 0$.

2. By Lemma 3.4.1, we have $V' < 0$ for sufficiently large positive r^*.

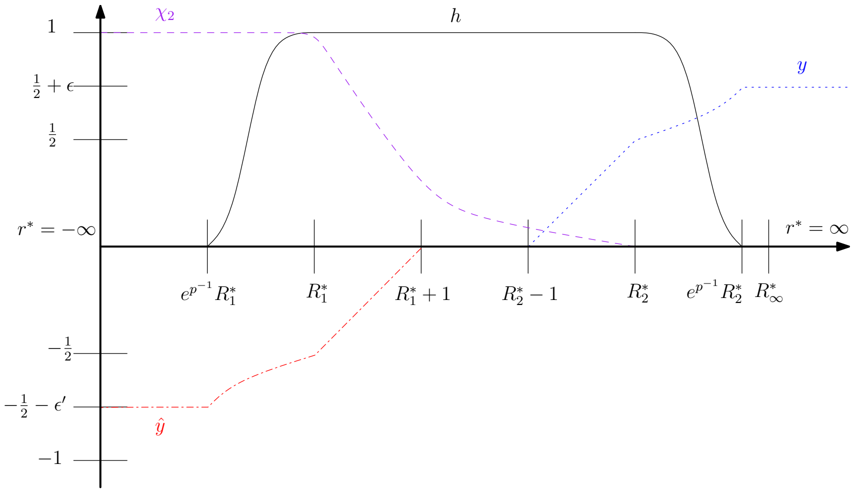

Figure 3.5.3: The functions χ_2, h, y and $\hat{y}$ used in the proof of Proposition 3.5.5.

3. It follows from (3.4.4) and (3.4.5) and the smoothness of V that for sufficiently small $\tilde{K}_0$, we have $V' > 0$ for r near r_+.

Now following the argument given in [DRSR14, §8.7.1], we may construct a current from the templates (3.3.26), (3.3.27), (3.3.17) and (3.3.23):

$$Q_{\flat,1,small} = Q_1^h + Q_2^y + Q_2^{\hat{y}} - Ey(\infty)Q_T + \chi_2 Q_K.$$

Here, Q_1^h is used to obtain a coercive estimate in a bounded interval $[R_1, R_2]$, with R_1 bounded away from the horizon and R_2 bounded. This achieved by cutting off an indicator function and introduces negative terms in the regions $[R_0, R_1]$ and $[R_2, R_3]$. The currents Q_2^y and $Q_2^{\hat{y}}$ are constructed to remedy this.

It then remains to deal with the boundary terms. Choosing $\tilde{K}_0$ small enough, we see from (3.3.19) that this frequency regime is non-superradiant. Thus the subtraction of the current $Ey(\infty)Q_T$ controls the boundary term at $r^* = \infty$. The $\chi_2 Q_K$ current is introduced to absorb the boundary term at $r^* = -\infty$. The function χ_2 is a smooth bounded cut-off that is identically 1 in $(-\infty, R_1]$ and identically 0 in $[R_2, \infty)$. Applying the $\chi_2 Q_K$ current gives control over the boundary term. Moreover, the bulk term $\int_{-\infty}^{\infty}(\chi_2 Q_K)'$ is supported only in $[R_1, R_2]$ and comes with a ω-weight. Since we already have a coercive estimate in $[R_1, R_2]$, we may absorb this term into the left hand side. See Figure 3.5.3 for the form of the functions used in constructing the currents. □

The near-stationary subrange (axisymmetric case): $|\omega| \leq \omega_0$ **and** $m = 0$

Proposition 3.5.6. *Let* $\omega \in \Omega$ *and* $(\omega, m, \Lambda) \in \mathcal{F}_{\flat,1}$ *and suppose* $m = 0$. *Let* u *be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16). *Then for all* $\omega_1 > 0$, $\lambda_2 > 0$, *sufficiently small* $\omega_0 > 0$ *sufficiently large* $R_\infty > R_e$ *and* $E > 2$, *there exist functions* y, $\hat{y}$ *and* h, *satisfying the uniform bounds*

$$|y| + |\hat{y}| + |h| \leq B,$$

$$y = 1,\ \hat{y} = 0,\ h = 0 \text{ for } r^* \geq R^*_\infty,$$

such that

$$\begin{aligned} &b\int_{R^*_-}^{R^*_+} \left(\left|u'\right|^2 + |u|^2\right) dr^* + b(\left|u'\right|^2 + \omega^2|u|^2)|_{r=\infty} \\ &\leq -\int_{-\infty}^{\infty} \left(2(y+\hat{y})\, Re(u'\overline{H}) + h\, Re(u\overline{H}) + E\omega Im(H\overline{u})\right). \end{aligned} \tag{3.5.7}$$

Proof. The properties of the potential V used to obtain (3.5.6) also hold here:

1. It is clear from (3.4.1) and (3.4.2) that if $m = 0$, for every $-\infty < \alpha < \beta < \infty$, we may take ω_0 small enough that $r \in [\alpha, \beta] \Rightarrow V - \omega^2 > 0$.

2. By Lemma 3.4.1, we have $V' < 0$ for sufficiently large positive r^*.

3. There is no superradiance in the axisymmetric case, so it follows directly from Lemma 3.4.3 and the smoothness of V that for sufficiently small $\tilde{K}_0$, we have $V' > 0$ for r near r_+.

The arguments from the proof of the estimate (3.5.6) may now be applied. The situation is simpler here: since there is no superradiance we do not need χ_2. The result follows from applying the following current, contructed from the templates (3.3.26), (3.3.27), (3.3.17):

$$Q_{\flat,1,m=0} = Q_1^h + Q_2^y + Q_2^{\hat{y}} - Ey(\infty)Q_T,$$

where E, h, y and $\hat{y}$ are as before. □

The near-stationary subrange (non-vanishing parameters case): $|\omega| \leq \omega_0$**,** $m \neq 0$ **and** $a^2 + Q^2 \geq \tilde{K}_0^2$

In this frequency regime, we exploit the non-vanishing of the parameters $m \neq 0$ and $0 < \tilde{K}_0^2 \leq a^2 + Q^2$.

Proposition 3.5.7. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\flat,1}$ and suppose $m \neq 0$ and $0 < \tilde{K}_0^2 \leq a^2 + Q^2$. Let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16)*,. Then for all $\omega_1 > 0$, $\lambda_2 > 0$, sufficiently small $\omega_0 > 0$ (depending on $\tilde{K}_0$), sufficiently large $R_\infty > R_e$ and $E > 2$, there exist functions y, $\hat{y}$ and h, satisfying the uniform bounds*

$$|\tilde{y}| + |y| + |h| + |\chi_1| + |\chi_2| \leq B\left(\tilde{K}_0\right),$$
$$and \quad |\tilde{y}| \leq B \exp\left(-br\right), \; y = 1, \; h = 0 \; for \; r^* \geq R^*_\infty,$$

such that

$$\begin{aligned} b\left(\tilde{K}_0\right) \int_{r_e^*}^{R_e^*} \left(|u'|^2 + |u|^2\right) dr^* + b(\left|u'\right|^2 + \omega^2|u|^2)|_{r=\infty} \\ \leq \int_{-\infty}^{\infty} \left(-2\tilde{y}\, Re(u'\overline{H}) - h\, Re(u\overline{H}) - 2y\, Re(u'\overline{H})\right) \; dr^* \\ + \int_{-\infty}^{\infty} \left(-E\chi_2\omega\, Im(H\overline{u}) - 2\chi_1\left(\omega - \omega_+\right)\, Im(H\overline{u})\right) \; dr^*. \end{aligned} \tag{3.5.8}$$

Proof. It suffices to show that we can adapt the argument given in [DRSR14, §8.7.2].

This amounts to verifying that

1. Since $m \neq 0$, there exists a constant $b = b(\tilde{K}_0) > 0$ such that

$$(\omega - \omega_+)^2 = \left(\omega - \frac{am}{2Mr_+ - Q^2}\right)^2 > b,$$

   provided we choose ω_0 small enough.

2. Since $m \neq 0$, $\Lambda \geq 2$. So

$$V = \Lambda r^{-2} + O(r^{-3}) \;\; as \;\; r \to \infty.$$

   This in turn implies that for every $1 \ll \alpha \ll \beta < \infty$, we may take ω_0 small enough that $r \in [\alpha, \beta] \Rightarrow V - \omega^2 > br^{-2}$. This positivity allows for the use of a Q_1^h current.

Armed with these properties of the potential, we can construct the following current from the templates (3.3.26), (3.3.27), (3.3.23) and (3.3.17):

$$Q_{\flat,1,non-vanishing} = Q_1^h + Q_2^{\tilde{y}} + Q_2^y - \chi_1 Q_K - E\chi_2 Q_T$$

for suitable functions h, $\tilde{y}$, y, χ_1 and χ_2 as given in [DRSR14, §8.7.2]. Refer to Figure 3.5.4 for the forms of these functions.

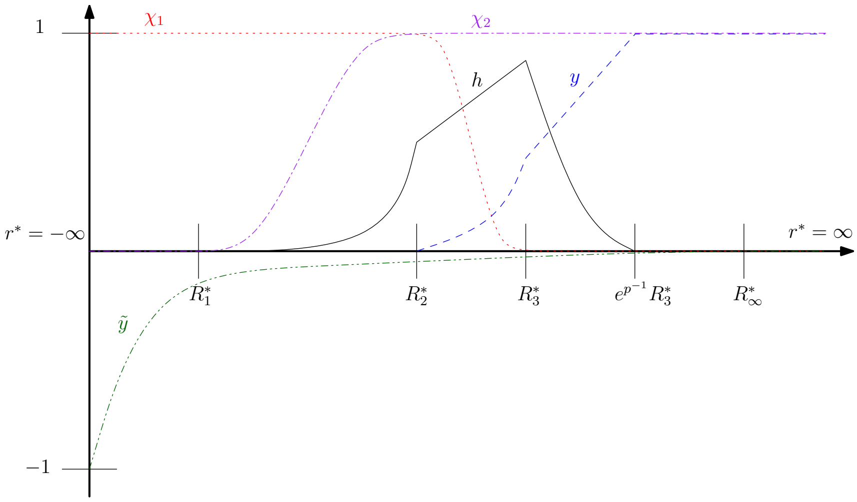

Figure 3.5.4: The functions χ_1, χ_2, h, y and $\tilde{y}$ used in the proof of Proposition 3.5.7.

Application of the $Q_2^{\tilde{y}}$ current yields the coercive term in the estimate. A large parameter is needed to deal with the boundary term at $r = r_+$. The Q_1^h current is deployed to provide this large parameter. The boundary term can then be dealt with by application of $\chi_1 Q_K$.

The cost of this construction is that it produces error terms that must then be absorbed by application of the Q_2^y current.

Finally, the boundary term at $r = \infty$ is handled by subtracting $E\chi_2 Q_T$ with $E \geq 2$. $\square$

Remark The function $\tilde{y}$ is defined by

$$\tilde{y}(r^*) := -\exp\left(-C\int_{-\infty}^{r^*} \upsilon dr^*\right), \tag{3.5.9}$$

where $\upsilon(r)$ is a positive function such that

$$\upsilon = \Delta \text{ near } r_+, \qquad \upsilon = 1 \text{ when } r^* \geq R_\infty^*, \qquad |\upsilon| \leq B.$$

Note that $\tilde{y}\,(-\infty) = -1$ and $\tilde{y}\,(\infty) = 0$. In particular $\tilde{y} < 0$ and $\tilde{y}' \neq 0$ in the $r \geq R_\infty$ range.

This differs from all our other seed functions so the error term $\int_{-\infty}^{\infty} 2\tilde{y}\,\mathrm{Re}(u'\overline{H})$ generated by $Q_2^{\tilde{y}}$ must be handled separately from the other error terms, see §3.6.3.

The nonstationary subrange: $|\omega| \geq \omega_0$

The estimate in this final frequency range is relatively simple aside from the presence of the horizon term $\left(|\omega(\omega-\omega_+)||u|^2\right)_{r=r_+}$ in (3.5.10). This term arises due to superradiant frequencies in the nonstationary bounded frequency regime $\mathcal{F}_{\flat,2}$. There are no known localised energy currents for dealing with these superradiant frequencies. As such, this horizon term gives rise to a term $1_{\{\omega_0\leq|\omega|\leq\omega_1\}\cap\left\{\Lambda\leq\lambda_2^{-1}\omega_1^2\right\}}\left|u^{(a\omega)}_{m\ell}(-\infty)\right|^2$ on the right hand side of (3.5.11) in the statement of Proposition 3.5.9. This troublesome term is controlled by applying the quantitative mode stability result of Chapter 4 after summation, see §3.6.

Proposition 3.5.8. *Let $\omega \in \Omega$ and $(\omega, m, \Lambda) \in \mathcal{F}_{\flat,2}$ and let u be a smooth solution of* (3.3.12) *with boundary conditions* (3.3.15) *and* (3.3.16)*. Then for all $\omega_1 > 0$, $\lambda_2 > 0$, sufficiently small $\omega_0 > 0$ (depending on $\tilde{K}_0$), sufficiently large $R_\infty > R_e$ and $E > 2$, there exists a function y satisfying the uniform bounds*

$$|y| \leq B, \quad \textit{and} \quad y = 1 \textit{ for } r^* \geq R^*_\infty,$$

such that

$$\begin{aligned} b\left(\omega_0,\omega_1\right)&\int_{r_e^*}^{R_e^*}\left(|u'|^2+|u|^2\right)dr^* + b(\left|u'\right|^2+\omega^2|u|^2)|_{r=\infty} \\ &\leq B\left(|\omega(\omega-\omega_+)||u|^2\right)_{r=r_+} - \int_{-\infty}^{\infty}\left(2y\, Re(u'\overline{H}) - E\omega Im(\overline{H}u)\right). \end{aligned} \tag{3.5.10}$$

Proof. The argument of [DRSR14, §8.7.4] applies directly. We apply a current of the form

$$Q_{\flat,2} = Q_2^y - EQ_T$$

where Q_2 and Q_T are defined by (3.3.27), (3.3.17) repectively and

$$y(r^*) := \exp\left(-C\int_{r^*}^{\infty}\chi_{R^*_\infty}r^{-2}dr\right).$$

Here $C = C(\omega_0,\omega_1,\lambda_2)$ is sufficiently large and the function $\chi_{R^*_\infty}$ is smooth, which identically 1 on $[r_+, R_\infty - 1)$ and identically 0 on $[R_\infty, \infty)$. Note that $y|_{r^*\geq R^*_\infty} = 1$ and $y\left(-\infty\right) = 0$. □

3.5.7 The general frequency-localised estimate

Note that the partitioning parameters λ_1, λ_2, ω_0 and ω_1 have been fixed in the proofs contained in §3.5.2 to §3.5.6. Observe that these proofs all hold for *all* sufficiently large but finite λ_1 and *all* sufficiently small postive λ_2. Let ω_1 be fixed and let $\lambda_1^* = \lambda_1$ and

$\lambda_2^* = \lambda_2$. If it is already the case that we require λ_1 so large that $\lambda_1 > (\lambda_2^*)^{-1}\omega_1$, we make a new choice of $\lambda_2 < \lambda_2^*$ to fulfil (3.5.1). In the case that we require λ_2 small enough that $\lambda_1^* < \lambda_2^{-1}\omega_1$, we finalise the choice of λ_1 by enlarging it so that (3.5.1) is satisfied.

Proposition 3.5.9. *Under the hypotheses of Theorem 3.3.2, there exist frequency independent positive constants b and E and bounded positive functions $C_{m\ell}^{(a\omega)}(r^*)$, $D_{m\ell}^{(a\omega)}(r^*)$ and $J_{m\ell}^{(a\omega)}(r^*)$ such that for all frequency triples (ω, m, Λ) where $\omega \in \Omega$, the following holds for each $u_{m\ell}^{(a\omega)}$ satisfying* (3.3.12) *and the boundary conditions* (3.3.15) *and* (3.3.16)*:*

$$\begin{aligned} & b\int_{r_e^*}^{R_e^*}\left(\left|\frac{d}{dr^*}u_{m\ell}^{(a\omega)}\right|^2 + \left|u_{m\ell}^{(a\omega)}\right|^2 + \chi_{m\ell}^{(a\omega)}(r)(\omega^2+\Lambda)\left|u_{m\ell}^{(a\omega)}\right|^2\right) dr^* + b\omega^2|u|^2|_{r=\infty} \\ \leq\ & \chi_{\flat,\text{☼}}\left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2 + \left|\int_{r_+}^{\infty}[C_{m\ell}^{(a\omega)}(r^*) - 2\tilde{y}]\mathrm{Re}(u'\bar{H}) + D_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u\bar{H})\ dr^*\right| \\ & + \left|\int_{r_+}^{\infty} J_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u' + i(\omega-\omega_+)u\bar{H}) + E\omega\mathrm{Im}\left[u\bar{H}\right]\ dr^*\right|, \end{aligned} \tag{3.5.11}$$

where

$$\begin{aligned} \chi_{\flat,\text{☼}}(\omega,m,\Lambda) &= 1_{\{\omega_0\leq|\omega|\leq\omega_1\}\cap\left\{\Lambda\leq\lambda_2^{-1}\omega_1^2\right\}}, \\ \chi_{m\ell}^{(a\omega)}(r) &= \begin{cases} (r-r_{max}^{(\omega,m,\ell)})^2 & \text{for each } (\omega,m,\Lambda)\in\mathcal{F}_\natural, \\ 1 & \text{for } (\omega,m,\Lambda)\notin\mathcal{F}_\natural \end{cases}, \\ C_{m\ell}^{(a\omega)}(r^*) &= B\left(|y|+|f|+1\right), \\ D_{m\ell}^{(a\omega)}(r^*) &= B\left(\left|f'+Ah\right|+\left|f'\right|+|h|\right) \\ \text{and}\quad J_{m\ell}^{(a\omega)}(r^*) &= B\Gamma\left|\frac{\zeta}{\tilde{V}}\right|, \end{aligned}$$

where B is a large positive constant.

Proof. We choose E and R_∞ large enough and $b > 0$ small enough that (3.5.2), (3.5.3), (3.5.4), (3.5.5), (3.5.6), (3.5.7), (3.5.8) and (3.5.10) all hold. Since every (ω, m, Λ) lies in one of the frequency ranges for which we have a frequency localised estimate for $\omega \in \Omega$, this establishes (3.5.11). The choice of E finalises the choice of the constants A and Γ in (3.5.2) and (3.5.5).

All of the frequency dependent functions f, h, y and ζ are bounded by the functions $C_{m\ell}^{(a\omega)}$, $D_{m\ell}^{(a\omega)}$ and $J_{m\ell}^{(a\omega)}$. The precise degeneration of the current for trapped frequencies was used to obtain $\chi_{m\ell}^{(a\omega)}$. The horizon term $\chi_{\flat,\text{☼}}\left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2$ arises from the superradiant frequencies in the bounded regime $\mathcal{F}_{\flat,2}$, see (3.5.10). □

3.6 Proof of the conditional (ILED)

The time has come to return to physical space. We turn our frequency-localised estimate (3.5.11) into a physical space estimate. We do this by summing the frequency localised estimate (3.5.11) over m and ℓ, integrating over ω and appealing to the Parseval-type identities of §3.3.3. Since these identities hold in $L^2(dw)$ it suffices that we proved the frequency localised estimates for almost every ω (see §3.3.3).

3.6.1 The physical space estimate

Proposition 3.6.1. *Let ψ be a solution of* (2.2.3) *arising from smooth, compactly supported data on Σ_0. Assume that $\psi_{\text{✂}}$ defined by* (3.3.3) *satisfies* (3.3.1) *and* (3.3.2). *Then there exist frequency independent constants $b > 0$ and $B > E$ such that for any time $\tau > 0$, (including the limit $\tau \to \infty$),*

$$
\begin{aligned}
& b\int_0^\tau \int_{\Sigma_t\cap[r_+,R_e^*]} \left[(\partial_{r^*}\psi_{\text{✂}})^2 + \psi_{\text{✂}}^2 + \chi_\natural\left((T\psi_{\text{✂}})^2 + (\not\nabla\psi_{\text{✂}})^2\right)\right]\, dt^* + b\int_{\mathcal{I}^+} \mathbb{J}^T[\psi_{\text{✂}}]\cdot n_{\mathcal{I}^+} \\
&\leq \left|\int_{-\infty}^{\infty}\int_{r_e}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} [C_{m\ell}^{(a\omega)}(r^*) - 2\tilde{y}]\mathrm{Re}(u'\bar{H}) + D_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u\bar{H})\, dr^* d\omega\right| \\
&\quad + \left|\int_{-\infty}^{\infty}\int_{r_e}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} J_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u' + i(\omega-\omega_+)u\bar{H})\, dr^* d\omega\right| \\
&\quad + B\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \omega\mathrm{Im}\left[u_{m\ell}^{(a\omega)}\bar{H}_{m\ell}^{(a\omega)}\right]\, dr^* d\omega\right| + B\int_{\Sigma_0} \mathbb{J}_\mu^N[\psi]n_{\Sigma_0}^\mu.
\end{aligned}
\tag{3.6.1}
$$

The functions $C_{m\ell}^{(a\omega)}(r^*)$, $D_{m\ell}^{(a\omega)}(r^*)$ and $J_{m\ell}^{(a\omega)}(r^*)$ are as in Proposition 3.5.9 and

$$
\chi_\natural(r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \chi_{m\ell}^{(a\omega)}(r)\cdot S_{m\ell}^{(a\omega)}(\cos\theta)e^{im\phi}e^{-i\omega t}\, d\omega \tag{3.6.2}
$$

in the $L^2(d\omega)\ell^2(m,\ell)$ sense.

Proof. First note that

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_t \cap [r_e, R_e^*]} \left[(\partial_{r^*}\psi_{\text{✂}})^2 + \psi_{\text{✂}}^2 + \chi_\natural \left((T\psi_{\text{✂}})^2 + (\slashed{\nabla}\psi_{\text{✂}})^2 \right) \right] \, dt^* \\
\leq\ &\int_{-\infty}^\infty \int_{\Sigma_t \cap [r_e, R_e^*]} \left[(\partial_{r^*}\psi_{\text{✂}})^2 + \psi_{\text{✂}}^2 + \chi_\natural \left((T\psi_{\text{✂}})^2 + (\slashed{\nabla}\psi_{\text{✂}})^2 \right) \right] \, dt^*.
\end{aligned}$$

Applying the physical space–Fourier space identities in §3.3.3, we obtain for any $\tau > 0$

$$\begin{aligned}
&\int_{-\infty}^\infty \int_{\Sigma_t \cap [r_e, R_e^*]} \left[(\partial_{r^*}\psi_{\text{✂}})^2 + \psi_{\text{✂}}^2 + \chi_\natural \left((T\psi_{\text{✂}})^2 + (\slashed{\nabla}\psi_{\text{✂}})^2 \right) \right] \, dt^* \\
=\ &\int_{-\infty}^\infty \int_{\mathbb{S}^2} \int_{r_e}^{R_e} \left[(\partial_{r^*}\psi_{\text{✂}})^2 + \psi_{\text{✂}}^2 + \chi_\natural \left((T\psi_{\text{✂}})^2 + (\slashed{\nabla}\psi_{\text{✂}})^2 \right) \right] \, \rho^2 \sin\theta \, d\theta \, d\phi \, dr \, dt^* \\
=\ &\int_{-\infty}^\infty \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \int_{r_e^*}^{R_e^*} \left| \frac{d}{dr^*} u_{m\ell}^{(a\omega)} \right|^2 + \left| u_{m\ell}^{(a\omega)} \right|^2 + \chi_{m\ell}^{(a\omega)}(r)(\omega^2 + \Lambda) \left| u_{m\ell}^{(a\omega)} \right|^2 dr^* d\omega
\end{aligned} \tag{3.6.3}$$

and since $\partial_t \gamma$ is not supported on $\mathcal{I}^+$,

$$\int_{\mathcal{I}^+} \mathbb{J}^T[\psi] \cdot n_{\mathcal{I}^+} = \int_{\mathcal{I}^+} \mathbb{J}^T[\psi_{\text{✂}}] \cdot n_{\mathcal{I}^+} = \int_{-\infty}^\infty \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \omega^2 \left| u_{m\ell}^{(a\omega)}(r = \infty) \right|^2.$$

Applying (3.5.11) for $\omega \in \Omega$, summing over m and ℓ and integrating with respect to ω, we have

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \int_{r_e^*}^{R_e^*} \left|\frac{d}{dr^*}u_{m\ell}^{(a\omega)}\right|^2 + \left|u_{m\ell}^{(a\omega)}\right|^2 + \chi_{m\ell}^{(a\omega)}(r)(\omega^2+\Lambda)\left|u_{m\ell}^{(a\omega)}\right|^2 dr^* d\omega \\
&\quad + \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \omega^2 |u|^2_{r=\infty} d\omega \\
&\leq b^{-1}\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} [C_{m\ell}^{(a\omega)} - 2\tilde{y}]\mathrm{Re}(u'\bar{H}) + D_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u\bar{H})\ dr^* d\omega\right| \\
&\quad + b^{-1}\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} J_{m\ell}^{(a\omega)}\mathrm{Re}(u' + i(\omega-\omega_+)u\bar{H})\ dr^* d\omega\right| \\
&\quad + \ \ b^{-1}B\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \omega \mathrm{Im}\left[u_{m\ell}^{(a\omega)}\bar{H}_{m\ell}^{(a\omega)}\right]\ dr^* d\omega\right| \\
&\quad + B\int_{\{\omega_0\leq|\omega|\leq\omega_1\}} \sum_{\substack{|m|(|m|+1)\leq\Lambda\\ \Lambda\leq\lambda_2^{-1}\omega_1^2}} \left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2 d\omega.
\end{aligned}
$$

Recall that we take $B > E$, where E is as in Proposition 3.5.9. The last term on the right hand side is controlled by application of the quantitative mode stability result (Theorem 4.5.1). That is, by Theorem 4.8.2,

$$
\int_{\{\omega_0\leq|\omega|\leq\omega_1\}} \sum_{\substack{|m|(|m|+1)\leq\Lambda\\ \Lambda\leq\lambda_2^{-1}\omega_1^2}} \left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2 d\omega \leq B_{\mathcal{F}_\flat}\int_{\Sigma_0} \mathbb{J}_\mu^N[\psi] n_{\Sigma_0}^\mu.
$$

Absorbing $B_{\mathcal{F}_\flat}$ into B, we have thus obtained

$$
\begin{aligned}
& b\int_0^\tau \int_{\Sigma_t\cap[r_e,R_e^*]} \left[(\partial_{r^*}\psi_{✂})^2+\psi_{✂}^2+\chi_\natural\left((T\psi_{✂})^2+(\slashed{\nabla}\psi_{✂})^2\right)\right]\, dt^* + b\int_{\mathcal{I}^+}\mathbb{J}^T[\psi]\cdot n_{\mathcal{I}^+} \\
&\leq \left|\int_{-\infty}^{\infty}\int_{r_e}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}[C_{m\ell}^{(a\omega)}-2\tilde{y}]\mathrm{Re}(u'\bar{H})+D_{m\ell}^{(a\omega)}\mathrm{Re}(u\bar{H})\ dr^*d\omega\right| \\
&\quad + \left|\int_{-\infty}^{\infty}\int_{r_e}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}J_{m\ell}^{(a\omega)}\mathrm{Re}(u'+i(\omega-\omega_+)u\bar{H})\ dr^*d\omega\right| \\
&\quad + B\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\omega\mathrm{Im}\left[u_{m\ell}^{(a\omega)}\bar{H}_{m\ell}^{(a\omega)}\right]\ dr^*d\omega\right| + B\int_{\Sigma_0}\mathbb{J}_\mu^N[\psi]n^\mu_{\Sigma_0}.
\end{aligned} \tag{3.6.4}
$$

The low frequency estimates do not give control all the way up to the horizon. We therefore couple the above estimate to an $\epsilon-$multiple of the red-shift estimate (2.2.6) with $R_0 = r_e$ and taking $\delta > 0$ small enough that $r_e+\delta < \inf_{\omega,m,\ell} r_{max}(\omega,m,\Lambda)$ allows us to extend the radial region of integration on the left hand side to $[r_+, R_e^*]$ and absorb last term on the right hand side of (2.2.6):

$$
\begin{aligned}
& bB_0^{-1}\epsilon\int_0^\tau\int_{\Sigma_t\cap[r_+,R_0]}\left[(\partial_{r^*}\psi_{✂})^2+\psi_{✂}^2+\chi_\natural\left((T\psi_{✂})^2+(\slashed{\nabla}\psi_{✂})^2\right)\right]\ \rho^2 dt^* \\
& +bB_0^{-1}(1-\epsilon)\int_0^\tau\int_{\Sigma_t\cap[R_0,R_e^*]}\left[(\partial_{r^*}\psi_{✂})^2+\psi_{✂}^2+\chi_\natural\left((T\psi_{✂})^2+(\slashed{\nabla}\psi_{✂})^2\right)\right]\ \rho^2 dt^* \\
& \quad +b\int_{\mathcal{I}^+}\mathbb{J}^T[\psi]\cdot n_{\mathcal{I}^+}+\epsilon\int_{\mathcal{H}^+(0,\tau)}\mathbb{J}^N_\mu[\psi]n^\mu_{\mathcal{H}^+} \\
\leq\ & \left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}[C_{m\ell}^{(a\omega)}-2\tilde{y}]\mathrm{Re}(u'\bar{H})+D_{m\ell}^{(a\omega)}\mathrm{Re}(u\bar{H})\ dr^*d\omega\right| \\
& +\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}J_{m\ell}^{(a\omega)}\mathrm{Re}(u'+i(\omega-\omega_+)u\bar{H})\ dr^*d\omega\right| \\
& +\ B\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\omega\mathrm{Im}\left[u_{m\ell}^{(a\omega)}\bar{H}_{m\ell}^{(a\omega)}\right]\ dr^*d\omega\right|+B\int_{\Sigma_0}\mathbb{J}_\mu^N[\psi]n^\mu_{\Sigma_0}.
\end{aligned}
$$

Taking ϵ small enough and absorbing B_0^{-1}, ϵ and $(1-\epsilon)$ into the constant b, we arrive at the the result. $\square$

Remark In proving the estimate above, we appealed to the quantitative mode stability results of Chapter 4. Note that appeal to mode stability is not necessary if we restrict to $a^2 + Q^2 \ll M^2$ or require that $m = 0$ or $m \gg 1$. In the former case superradiance is absent. In the latter case, the unfavourably signed boundary terms that arise due to superradiance may be dominated using the redshift current directly, see §1.5.

Proposition 3.6.2. *Let ψ be a solution of* (2.2.3) *arising from smooth, compactly supported data on Σ_0. Assume that $\psi_{\scissors}$ defined by* (3.3.3) *satisfies* (3.3.1) *and* (3.3.2)*. There exists a uniform constant $C_{R_e} > 0$ such that for any time $\tau > 0$, (including the limit $\tau \to \infty$),*

$$\begin{aligned} &\int_0^\tau \int_{\Sigma_t \cap [r_+, R_e^*]} \left[(\partial_{r^*}\psi_{\scissors})^2 + \psi_{\scissors}^2 + \chi_{\natural}\left((T\psi_{\scissors})^2 + (\not\nabla\psi_{\scissors})^2 \right) \right] \rho^2 dt^* \\ &\quad + \int_{\mathcal{I}^+} \mathbb{J}^T[\psi] \cdot n_{\mathcal{I}^+} + \int_{\mathcal{H}^+(0,\tau)} \mathbb{J}^N[\psi] \cdot n_{\mathcal{H}^+} \leq C_{R_e} \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}. \end{aligned} \tag{3.6.5}$$

Proof. It remains to control the right hand side of (3.6.1) by data. In order to control the terms containing $C^{(a\omega)}_{m\ell}$, $D^{(a\omega)}_{m\ell}$ and $J^{(a\omega)}_{m\ell}$, we take $R_\infty \gg R_e$ and split the integral in r^* into two regions

$$\mathcal{B} = \{r_+ \leq r \leq R_\infty\} \quad \text{and} \quad \mathcal{U} = \{r > R_\infty\}.$$

which we deal with separately. The integral over the compact region

$$\int_{-\infty}^{\infty} \int_{\mathcal{B}} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} [C^{(a\omega)}_{m\ell}(r^*) - 2\tilde{y}]\mathrm{Re}(u'\bar{H}) + D^{(a\omega)}_{m\ell}(r^*)\mathrm{Re}(u\bar{H}) + J^{(a\omega)}_{m\ell}\mathrm{Re}(u' + i(\omega-\omega_+)u\bar{H})\, dr^* d\omega$$

is controlled §3.6.2 and the integral over the unbounded region

$$\int_{-\infty}^{\infty} \int_{\mathcal{U}} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} [C^{(a\omega)}_{m\ell}(r^*) - 2\tilde{y}]\mathrm{Re}(u'\bar{H}) + D^{(a\omega)}_{m\ell}(r^*)\mathrm{Re}(u\bar{H}) + J^{(a\omega)}_{m\ell}\mathrm{Re}(u' + i(\omega-\omega_+)u\bar{H})\, dr^* d\omega$$

is controlled in §3.6.3.

The term

$$\int_{-\infty}^{\infty} \int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \omega \mathrm{Im}\left[u^{(a\omega)}_{m\ell} \bar{H}^{(a\omega)}_{m\ell} \right] dr^* d\omega$$

is dealt with in §3.6.4.

3.6.2 Error terms in $\mathcal{B} = \{r_+ \leq r \leq R_\infty^*\}$

In this section we control the following error term by data

$$\int_{-\infty}^{\infty}\int_{\mathcal{B}}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}[C_{m\ell}^{(a\omega)}+J_{m\ell}^{(a\omega)}-2\tilde{y}]\mathrm{Re}(u'\bar{H})+\mathrm{Re}\left[\left(D_{m\ell}^{(a\omega)}(r^*)+i(\omega-\omega_+)J_{m\ell}^{(a\omega)}\right)u\bar{H}\right]\ dr^*d\omega.$$

In this region we want to absorb error terms into the left hand side of (3.6.1). For $r^*\in\mathcal{B}$:

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}[C_{m\ell}^{(a\omega)}+J_{m\ell}^{(a\omega)}-2\tilde{y}]\mathrm{Re}(u'\bar{H})+\mathrm{Re}\left[\left(D_{m\ell}^{(a\omega)}+i(\omega-\omega_+)J_{m\ell}^{(a\omega)}\right)u\bar{H}\right]\ d\omega\right|\\
\leq\ &\varepsilon^{-1}\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\left[(C_{m\ell}^{(a\omega)}(r^*))^2+(D_{m\ell}^{(a\omega)}(r^*))^2+(J_{m\ell}^{(a\omega)}(r^*))^2+4\tilde{y}(r^*)^2\right](H_{m\ell}^{(a\omega)})^2\ d\omega\\
&+\varepsilon\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\left([(u_{m\ell}^{(a\omega)})']^2+(u_{m\ell}^{(a\omega)})^2+1_{\mathcal{F}_{☼}}(\omega-\omega_+)^2(u_{m\ell}^{(a\omega)})^2\right)\ d\omega.
\end{aligned}$$

Taking the physical/Fourier space identities in §3.3.3 into account, we have

$$\begin{aligned}
&\varepsilon\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\left([(u_{m\ell}^{(a\omega)})']^2+(u_{m\ell}^{(a\omega)})^2+1_{\mathcal{F}_{☼}}(\omega-\omega_+)^2(u_{m\ell}^{(a\omega)})^2\right)\ d\omega\\
\leq\ &\varepsilon C\int_0^{\infty}\int_{\mathbb{S}^2}\frac{\Delta}{r^2+a^2}[(r^2+a^2)(\partial_{r^*}\psi_{✂})^2+\check{1}_{\mathcal{F}_{☼}}\partial_t\psi_{✂}^2+\psi_{✂}^2]\ \ dt\ dg_{\mathbb{S}^2}.
\end{aligned}$$

Integrating this term over r^* in the region $\mathcal{B}$ and taking ε small enough, this term can be absorbed into the left hand side of (3.6.1). Note that this works for the $\partial_t\psi_{✂}$ term as it is only supported in the superradiant regime, where (3.6.1) does not degenerate due to trapping.

Recall from Proposition 3.5.9 that the functions $C_{m\ell}^{(a\omega)}(r^*)$, $D_{m\ell}^{(a\omega)}(r^*)$, $J_{m\ell}^{(a\omega)}(r^*)$ and $\tilde{y}(r^*)$ are bounded (uniformly w.r.t (r,ω,m,ℓ)) in the compact region $\mathcal{B}$, so

$$\begin{aligned}
&\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\varepsilon^{-1}\left[(C_{m\ell}^{(a\omega)}(r^*))^2+(D_{m\ell}^{(a\omega)}(r^*))^2+(J_{m\ell}^{(a\omega)}(r^*))^2+(\tilde{y}(r^*))^2\right](H_{m\ell}^{(a\omega)})^2\ d\omega\\
&\leq C\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\varepsilon^{-1}\frac{\Delta^2}{r^2+a^2}(F_{m\ell}^{(a\omega)})^2\ d\omega\\
&\leq C\int_0^1\int_{\mathbb{S}^2}\varepsilon^{-1}\frac{\Delta^2}{r^2+a^2}F^2\ \ dt\ dg_{\mathbb{S}^2}.
\end{aligned}$$

Now integrating over r^* in the region $\mathcal{B}$ and recalling the definition of F we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\int_{r_+}^{R_\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\varepsilon^{-1}\left[(C_{m\ell}^{(a\omega)})^2+(D_{m\ell}^{(a\omega)})^2+(J_{m\ell}^{(a\omega)})^2+(\tilde{y}(r^*))^2\right]\left|H_{m\ell}^{(a\omega)}\right|^2\,d\omega dr^*\\
\leq\;& C\varepsilon^{-1}\int_0^1\int_{r_+}^{R_\infty}\int_{\mathbb{S}^2}\frac{\Delta^2}{r^2+a^2}((\Box\gamma)\psi+2\nabla^\mu\gamma\nabla_\mu\psi)^2\;\;dt^*\;dg_{\mathbb{S}^2}dr^*\\
\leq\;& C\varepsilon^{-1}\int_0^1\int_{r_+}^{R_\infty}\int_{\mathbb{S}^2}\frac{\Delta^2}{r^2+a^2}[(\Box\gamma)^2\psi^2+|\nabla\gamma|^2|\nabla\psi|^2]\;\;dt^*\;dg_{\mathbb{S}^2}dr^*\\
\leq\;& C\varepsilon^{-1}\int_0^1\int_{\Sigma_t}\mathbb{J}_\mu^N[\psi]n_{\Sigma_t}^\mu\;dt^*.
\end{aligned}$$

Here we have used the Hardy inequality (2.2.8) in r to control the term containing ψ^2. It now follows from Proposition 3.3.1 that

$$\int_0^1\int_{\Sigma_0}\mathbb{J}_\mu^N[\psi]n_{\Sigma_t}^\mu\;dt^*\quad\leq\quad e^P\int_{\Sigma_0}\mathbb{J}_\mu^N[\psi]n_{\Sigma_0}^\mu.$$

We have thus controlled all error terms in the bounded region $\mathcal{B}$.

3.6.3 Error terms in $\mathcal{U}=\{r\geq R_\infty^*\}$

We now control the term

$$\int_{-\infty}^{\infty}\int_{\mathcal{U}}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}[C_{m\ell}^{(a\omega)}(r^*)-2\tilde{y}]\mathrm{Re}(u'\bar{H})+D_{m\ell}^{(a\omega)}(r^*)\mathrm{Re}(u\bar{H})+J_{m\ell}^{(a\omega)}\mathrm{Re}(u'+i(\omega-\omega_+)u\bar{H})\;dr^*d\omega.$$

Recall that

$$\begin{aligned}
C_{m\ell}^{(a\omega)}(r^*)&=B\left(|y|+|f|+1\right),\\
D_{m\ell}^{(a\omega)}(r^*)&=B\left(|f'+Ah|+|f'|+|h|\right)\\
\text{and}\quad J_{m\ell}^{(a\omega)}(r^*)&=B\Gamma\left|\frac{\zeta}{\tilde{V}}\right|,
\end{aligned}$$

Looking back to §3.5, we see that $\zeta=h=y=f'=0$ and $f=1$ in $\mathcal{U}$. Therefore, $C_{m\ell}^{(a\omega)}(r^*)$ is constant and $D_{m\ell}^{(a\omega)}(r^*)=J_{m\ell}^{(a\omega)}(r^*)=0$ in $\mathcal{U}$. It thus remains only to estimate the term $(B-2\tilde{y})\mathrm{Re}(u'\bar{H})$ in $\mathcal{U}$.

Recall the definition of the inhomogeneity

$$F=(\Box\gamma)\psi+2\nabla^\mu\gamma\nabla_\mu\psi.$$

For the term containing the function $\tilde{y}$, we note that y is not constant in $\mathcal{U}$ (see (3.5.9)) but since $|\tilde{y}| \leq \exp(-br^*)$ as $r^* \to \infty$, we may apply Plancherel and a Hardy inequality in r^* to obtain

$$
\begin{aligned}
&\left| \int_{\{\omega_0 \leq |\omega| \leq \omega_1\}} \sum_{\substack{|m|(|m|+1)\leq\Lambda \\ \Lambda\leq\lambda_2^{-1}\omega_1^2}} \int_{R^*_\infty}^{\infty} -2\tilde{y}\mathrm{Re}(u'\bar{H}) dr\, d\omega \right| \\
&= \left| \int_0^1 \int_{R^*_\infty}^{\infty} \int_{\mathbb{S}^2} -2\tilde{y} \cdot \mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2} \psi_{\text{✂}} \right) \overline{\Delta\left(r^2+a^2\right)^{-1/2} F} \right) \sin\theta\, dt\, dr^*\, d\theta\, d\phi \right| \\
&\leq B \int_0^1 \int_{\Sigma_t \cap [R^*_\infty, \infty)} \Delta \exp(-br^*) \left| \left(\gamma\partial_{r^*}\psi + r^{-1}(\gamma + \partial_r\gamma)\psi\right) \left((\Box\gamma)\psi + 2\nabla^\mu\gamma\nabla_\mu\psi\right) \right| dt \\
&\leq BC \int_0^1 \int_{\Sigma_t} \mathbb{J}^N[\psi] \cdot n_{\Sigma_t}\, dt \\
&\leq BCe^P \int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0},
\end{aligned}
$$

by Proposition 3.3.1.

Let us now deal with the term not containing $\tilde{y}$. By the physical/Fourier space identities in §3.3.3, and the support of γ,

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \int_{R^*_\infty}^{\infty} \sum_{m,\ell} \mathrm{Re}(u'\bar{H}) d\omega dr^* \\
= \quad &\int_{-\infty}^{\infty} \int_{R_\infty}^{\infty} \int_{\mathbb{S}^2} \mathrm{Re}\left(\partial_{r^*}\left((r^2+a^2)^{1/2} \psi_{\text{✂}} \right) \overline{\Delta (r^2+a^2)^{-1/2} F} \right) \ \sin\theta d\theta\ d\phi\ dr^*\ dt^*.
\end{aligned}
$$

Observe that for R_∞ large enough, γ depends only on t, so

$$
F = \left(r^2+a^2\right)^{-1} \rho^2 \left(2g^{tt}\partial_t\gamma\partial_t\psi + 2g^{t\phi}\partial_t\gamma\partial_\phi\psi + g^{tt}\partial_t^2\gamma\psi \right) \qquad \text{for } \ r \geq R_\infty.
$$

So (suppressing the factor $\sin\theta\, dt\, dr\, d\theta\, d\phi$),

$$
\begin{aligned}
&\int_0^\infty \int_{R^*_\infty}^{\infty} \int_{\mathbb{S}^2} \mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2} \psi_{\text{✂}} \right) \overline{\Delta\left(r^2+a^2\right)^{-1/2} F} \right) \\
&= \int_0^\infty \int_{R^*_\infty}^{\infty} \int_{\mathbb{S}^2} \mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2} \psi_{\text{✂}} \right) \overline{\Delta\left(r^2+a^2\right)^{-3/2} \rho^2 \left(2g^{tt}\partial_t\gamma\partial_t\psi + 2g^{t\phi}\partial_t\gamma\partial_\phi\psi\right)} \right) \\
&\quad + \int_0^\infty \int_{R^*_\infty}^{\infty} \int_{\mathbb{S}^2} \mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2} \psi_{\text{✂}} \right) \overline{\Delta\left(r^2+a^2\right)^{-3/2} \rho^2 g^{tt}\partial_t^2\gamma\psi} \right).
\end{aligned}
$$

For the $g^{tt}\partial_t\gamma\partial_t\psi$ term:

$$\begin{aligned}
&\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2}\psi_{\text{✂}}\right)\overline{\Delta\left(r^2+a^2\right)^{-3/2}\rho^2\left(2g^{tt}\partial_t\gamma\partial_t\psi\right)}\right)\right|\\
&\leq B\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\mathrm{Re}\left(\left(\partial_{r^*}\psi_{\text{✂}}\right)\overline{\Delta\left(r^2+a^2\right)^{-1}\rho^2\left(2g^{tt}\partial_t\gamma\partial_t\psi\right)}\right)\right|\\
&\quad+B\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\frac{r}{(r^2+a^2)^{1/2}}\mathrm{Re}\left(\left(\psi_{\text{✂}}\right)\overline{\Delta\left(r^2+a^2\right)^{-3/2}\rho^2\left(2g^{tt}\partial_t\gamma\partial_t\psi\right)}\right)\right|\\
&\leq Be^P\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_\tau},
\end{aligned}$$

where the last inequality follows by applying the Hardy inequality (2.2.8) and Proposition 3.3.1.

Since $g^{t\phi}=-a(2Mr-Q^2)\Delta^{-1}\rho^{-2}=O\left(r^{-3}\right)$, the $g^{t\phi}\partial_t\gamma\partial_\phi\psi$ term can dealt with in the same way.

Let $\chi_{\mathcal{B}}$ be a smooth cut-off which is identically 1 for $r\leq R_\infty-1$ and identically 0 for $r\geq R_\infty$. Then since $\partial_{r^*}\gamma=0$ for $r\geq R_e$,

$$\begin{aligned}
&\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2}\psi_{\text{✂}}\right)\overline{\Delta\left(r^2+a^2\right)^{-3/2}\rho^2g^{tt}\partial_t^2\gamma\psi}\right)\right|\\
&=\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\chi_{\mathcal{B}}\Delta\left(r^2+a^2\right)^{-2}\rho^2g^{tt}\partial_t^2\gamma\gamma\mathrm{Re}\left(\partial_{r^*}\left(\left(r^2+a^2\right)^{1/2}\psi\right)\overline{\left(r^2+a^2\right)^{1/2}\psi}\right)\right|\\
&=\frac{1}{2}\left|\int_0^\infty\int_{R^*_\infty}^\infty\int_{\mathbb{S}^2}\chi_{\mathcal{B}}\partial_{r^*}\left(\Delta\left(r^2+a^2\right)^{-2}\rho^2g^{tt}\partial_t^2\gamma\gamma\right)\left(r^2+a^2\right)|\psi|^2\right|\\
&\leq B\int_0^1\int_{\Sigma_\tau\cap[R_{e^*},\infty)}\frac{|\psi|^2}{r^2}\leq Be^P\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.
\end{aligned}$$

Again, the last inequality follows by applying the Hardy inequality (2.2.8) and Proposition 3.3.1. Combining everything implies

$$\left|\int_{R^*_\infty}^\infty\sum_{m,\ell}\left(\int_{R^*_\infty}^\infty 2\mathrm{Re}\left(u'\overline{H}\right)\right)d\omega\right|\leq Be^P\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.$$

3.6.4 Controlling the error from the conserved energy current

We first note that the constant B is frequency independent and that this term arises from the energy identity for $\mathbb{J}^T_\mu[\psi_{\text{✂}}]$. By Parseval's identity and expanding $\mathcal{E}^T$,

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}}\omega\mathrm{Im}\left[u^{(a\omega)}_{m\ell}\bar{H}^{(a\omega)}_{m\ell}\right]\,drd\omega\right|\\
=&\left|\int_{-\infty}^{\infty}\int_{r_+}^{\infty}\int_{\mathbb{S}^2}\mathrm{Re}[T(\gamma\psi)]F\Delta\,dg_{\mathbb{S}^2}drdt\right|\\
=&\left|\int_0^1\int_{r_+}^{\infty}\int_{\mathbb{S}^2}\mathcal{G}\mathrm{Re}\left(\partial_t\gamma\psi+\gamma\partial_t\psi\right)\,dg_{\mathbb{S}^2}drdt.\right|\\
\leq&\left|\int_0^1\int_{r_+}^{R_\infty}\int_{\mathbb{S}^2}\chi_{\mathcal{B}}\mathcal{G}\mathrm{Re}\left(\partial_t\gamma\psi+\gamma\partial_t\psi\right)\,dg_{\mathbb{S}^2}drdt\right| && (3.6.6)\\
&+\left|\int_0^1\int_{R_\infty-1}^{\infty}\int_{\mathbb{S}^2}(1-\chi_{\mathcal{B}})\mathcal{G}\mathrm{Re}\left(\partial_t\gamma\psi+\gamma\partial_t\psi\right)dg_{\mathbb{S}^2}drdt\right|, && (3.6.7)
\end{aligned}$$

were $\chi_{\mathcal{B}}$ is a smooth cut-off which is identically 1 for $r\leq R_\infty-1$ and identically 0 for $r\geq R_\infty$ and

$$\mathcal{G}=\frac{\Delta\rho^2}{r^2+a^2}\left(2g^{tt}\partial_t\gamma\partial_t\psi+2g^{t\phi}\partial_t\gamma\partial_\phi\psi+g^{tt}\partial_t^2\gamma\psi\right).$$

The integral over the bounded region, (3.6.6), can be controlled by data as in §3.6.2. The other term requires more care.

Most of the terms in (3.6.7) can be dealt with painlessly:

$$\begin{aligned}
&\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}g^{t\phi}\partial_t\gamma\mathrm{Re}\left((\partial_t\gamma\psi)\,\overline{\partial_\phi\psi}\right)\right|\\
&\quad=\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}g^{t\phi}\left(\partial_t\gamma\right)^2\partial_\phi\left|\psi\right|^2\right|=0.
\end{aligned}$$

$$\begin{aligned}
\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}g^{t\phi}\gamma\partial_t\gamma\mathrm{Re}\left((\partial_t\psi)\,\overline{\partial_\phi\psi}\right)\right|&\leq C\int_0^1\int_{\Sigma_\tau}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_\tau}\\
&\leq C\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.
\end{aligned}$$

$$2\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}g^{tt}\gamma\partial_t\gamma\left|\partial_t\psi\right|^2\right|\leq C\int_0^1\int_{\Sigma_\tau}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_\tau}\leq C\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.$$

It remains to deal with the following term:

$$\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}\left(\mathrm{Re}\left(\partial_t\gamma\psi\overline{\left(2g^{tt}\partial_t\gamma\partial_t\psi+g^{tt}\partial_t^2\gamma\psi\right)}\right)+\mathrm{Re}\left(\gamma\partial_t\psi\overline{g^{tt}\partial_t^2\gamma\psi}\right)\right)\right|$$
$$\leq\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}+\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}g^{tt}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}\left(\gamma\partial_t\gamma\mathrm{Re}\left(\partial_t^2\psi\overline{\psi}\right)\right)\right|.$$

See [DRSR14, §9.6] for the derivation of this inequality. Instead of additional integration by parts on the last term, we use that ψ solves the wave equation:

$$\begin{aligned}g^{tt}\partial_t^2\psi=&\frac{2a(2Mr-Q^2)}{\rho^2\Delta}\partial_\phi\partial_t\psi-\frac{\Delta-a^2\sin^2\theta}{\Delta\rho^2\sin^2\theta}\partial_\phi^2\psi\\&-\frac{r^2+a^2}{\Delta\rho^2}\partial_{r^*}\left(\left(r^2+a^2\right)\partial_{r^*}\psi\right)-\frac{1}{\rho^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\psi\right).\end{aligned}$$

Substituting the right hand side of the wave equation above for $g^{tt}\partial_t^2\psi$ and integrating by parts in ϕ, r and θ, we have

$$\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}g^{tt}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}\left(\gamma\partial_t\gamma\mathrm{Re}\left(\partial_t^2\psi\overline{\psi}\right)\right)\right|\leq B\int_0^1\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.$$

Now by Proposition 3.3.1,

$$\left|\int_0^1\int_{-\infty}^{\infty}\int_{\mathbb{S}^2}g^{tt}\frac{\Delta\rho^2(1-\chi_{\mathcal{B}})}{r^2+a^2}\left(\gamma\partial_t\gamma\mathrm{Re}\left(\partial_t^2\psi\overline{\psi}\right)\right)\right|\leq B\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.$$

This concludes the proof of Proposition 3.6.2, our integrated local energy decay result for $\psi_{✂}$. □

3.6.5 Concluding the proof of the conditional (ILED)

We now show that (ILED) holds for ψ from the result for $\psi_{✂}$.

Proposition 3.6.3. *For a solution ψ of (2.2.3) satisfying the hypotheses of Theorem 3.3.2 and for any time $\tau>0$, (including the limit $\tau\to\infty$),*

$$\begin{aligned}&\int_0^\tau\int_{\Sigma_t\cap[r_+^*,R_e^*]}\left[(\partial_{r^*}\psi)^2+\psi^2+\chi_\natural\left((T\psi)^2+|\nabla\!\!\!/\psi|^2\right)\right]dt^* &(3.6.8)\\&\quad+\int_{\mathcal{I}^+}\mathbb{J}^T[\psi]\cdot n_{\mathcal{I}^+}+\int_{\mathcal{H}^+(0,\tau)}\mathbb{J}^N[\psi]\cdot n_{\mathcal{H}^+}\\&\leq C\int_{\Sigma_0}\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}, &(3.6.9)\end{aligned}$$

where C depends only on M, r_e, R_e and P. Moreover, $\chi_\natural>0$ for $r\notin\{r_e\leq r\leq R_\natural<R_e\}$.

Proof. Let $I[\psi] = (\partial_{r^*}\psi)^2 + \psi^2 + \chi_\natural \left((T\psi)^2 + (\nabla\!\!\!/\psi)^2\right)$.

Recall that $S_\gamma := \{0 \le t^* \le 1\}$ so $\psi = \psi_{✂}$ in $\mathcal{R}(0,\tau) \setminus S_\gamma$,

$$\int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\setminus S_\gamma} I[\psi] = \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\setminus S_\gamma} I[\psi_{✂}].$$

By (3.6.5),

$$\begin{aligned}
\int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]} I[\psi] &= \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\setminus S_\gamma} I[\psi] + \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\cap S_\gamma} I[\psi] \\
&= \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\setminus S_\gamma} I[\psi_{✂}] + \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]\cap S_\gamma} I[\psi] \\
&\le \int_{\mathcal{R}(0,\tau)\cap[r_+^*,R_e^*]} I[\psi_{✂}] + \int_0^1 \int_{\Sigma_t} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_t}\, dt^* \\
&\le C \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0} + e^P \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0},
\end{aligned}$$

where we have applied Proposition 3.3.1. □

We have thus proved (3.3.6).

3.6.6 Integrated decay up to null infinity

We now extend (3.3.6) up to null infinity, consequently proving (3.3.7). We make use of the following energy estimate for large r.

Proposition 3.6.4. *Fix $M > 0$ and $a^2 + Q^2 \le K_0^2 < M^2$. Let ψ be a solution of (2.2.3) satisfying the hypotheses of Theorem 3.3.2 and $\psi_\infty = 0$. For any $\delta > 0$, there exist positive constants $B(\delta)$, $2M < R_0 < R_e$, such that for any time $\tau > 0$, (including the limit $\tau \to \infty$),*

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_s\cap\{r\ge R_e\}} r^{-1}(r^{-\delta}|\partial_r\psi|^2 + r^{-\delta}|\partial_t\psi|^2 + |\nabla\!\!\!/\psi|^2 + r^{-2-\delta}\psi^2)\, ds \\
&\quad \le B(\delta)\left(\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0} + \int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau}\right).
\end{aligned}$$

Proof. Following [DR11a, §6], we let $\delta > 0$ and apply the current $\mathbb{J}^{X,w}[\psi]$ with

$$\begin{aligned}
w &= 2f'(r^*) + 4\frac{r-2M}{r^2} f(r^*) - 2\delta\frac{r-2M}{r^{2+\delta}} f(r^*), \\
X &= f(r^*)\partial_r, \\
f(r^*) &= \chi(1 - r^{-\delta}),
\end{aligned}$$

where χ is a smooth cut-off such that $\chi = 1$ for $r \geq R_e$, $\chi = 0$ for $r \leq R_e - 1$. Then

$$\begin{aligned}\mathbb{K}^{X,w}[\psi] &= \left(\frac{r}{r-2M}f'(r^*) - \frac{f(r^*)\delta}{2r^{1+\delta}}\right)(\partial_{r^*}\psi)^2 + \frac{f(r^*)\delta}{2r^{1+\delta}}(\partial_t\psi)^2 \\ &\quad + \left(\frac{r-3M}{r^2} - \frac{\delta(r-2M)}{r^{2+\delta}}\right) f(r^*)|\nabla\!\!\!/\psi|^2 - \frac{1}{2}(\Box w)(\psi^2)\end{aligned}$$

Taking R_e large enough

$$\mathbb{K}^{X,w}[\psi] \geq b(\delta)\left(r^{-1-\delta}(\partial_{r^*}\psi)^2 + r^{-1-\delta}(\partial_t\psi)^2 + r^{-1}|\nabla\!\!\!/\psi|^2 + r^{-3-\delta}\psi^2\right) \qquad \text{for } r \geq R_e.$$

We now apply the energy identity between Σ_0 and Σ_τ. Since $\partial_r\chi$ is compactly supported and $R_e - 1 > R_\natural$, we can use (3.3.6) to control the spacetime error terms it generates. We must control the error term

$$\begin{aligned}&\int_0^\tau \int_{\Sigma_s} \mathcal{E}^{V,w}[\psi] ds \\ &= \int_0^\tau \int_{\Sigma_s \cap \{r \geq R_e - 1\}} (\chi(1-r^{-\delta})\partial_r\psi)F + \frac{1}{4}(w\psi F) \\ &= \int_0^1 \int_{\Sigma_s \cap \{r \geq R_e - 1\}} (\chi(1-r^{-\delta})\partial_r\psi + \frac{1}{4}w\psi)\left(2g^{tt}\partial_t\gamma\partial_t\psi + 2g^{t\phi}\partial_t\gamma\partial_\phi\psi + g^{tt}\partial_t^2\gamma\psi\right).\end{aligned}$$

The terms involving only first order derivatives of ψ can immediately be controlled using Proposition 3.3.1.

For zeroth order terms we first use a Hardy inequality in r and then Proposition 3.3.1. This can immediately be done for terms containing w as they come with the required weights in r^{-1}.

The remaining term is dealt with by integrating by parts:

$$\begin{aligned}&\int_0^1 \int_{\Sigma_s \cap \{r \geq R_e - 1\}} g^{tt}\chi(1-r^{-\delta})(\partial_t^2\gamma)(\psi\partial_r\psi) ds \\ &= \int_0^1 \int_{\Sigma_s \cap \{r \geq R_e - 1\}} \Big[g^{tt}(\partial_r\chi)(1-r^{-\delta})(\partial_t^2\gamma)\psi + \partial_r g^{tt}\chi(1-r^{-\delta})(\partial_t^2\gamma)\psi \\ &\qquad + g^{tt}\chi(1-r^{-\delta})(\partial_t^2\gamma)\partial_r\psi + g^{tt}\chi(\delta r^{-1-\delta})(\partial_t^2\gamma)\psi\Big](\psi).\end{aligned}$$

The first term is compactly supported so it can be controlled by (3.3.6). The other terms, apart from the last one all have sufficient weights in r^{-1} to apply Hardy inequalities and use Proposition 3.3.1. We integrate the final term by parts to obtain

$$\int_\infty^\infty \int_{\Sigma_s \cap \{r \geq R_e - 1\}} g^{tt}\chi(\delta r^{-1-\delta})(\partial_t^2\gamma).(\psi^2) = \int_0^1 \int_{\Sigma_s \cap \{r \geq R_e - 1\}} g^{tt}\chi(\delta r^{-1-\delta})(\partial_t\gamma)(\psi\partial_t\psi).$$

After Cauchy-Schwarz, this term has the required r-weight to apply Hardy inequalities and use Proposition 3.3.1. Finally, we use Hardy inequalities in r to control $\int_\Sigma \mathbb{J}^{X,w}[\psi]\cdot n_\Sigma \leq C\int_\Sigma \mathbb{J}^N[\psi]\cdot n_\Sigma$ on the spacelike hypersurfaces. □

Remark Recall from §3.3.1 that the assumption $\psi_\infty = 0$ can be made without loss of generality.

We must take care in applying Proposition 3.6.4 since we do not have an (NEB) result yet and the right hand side has an error term supported on Σ_τ. However, the assumption (3.3.1) implies that

$$\int_{\Sigma_\tau\cap[r_+,R_e]} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_\tau} \in L^1_\tau[0,\infty).$$

By the pigeon-hole principle, there exists a constant C and a dyadic sequence $\tau_n\to\infty$ such that

$$\int_{\Sigma_{\tau_n}\cap[r_+,R_e]} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_{\tau_n}} \leq \frac{C}{\tau_n}. \tag{3.6.10}$$

Since T is timelike in the region $r\geq R_e$ we may apply its associated energy estimate

$$\begin{aligned}\int_{\Sigma_{\tau_n}} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_{\tau_n}} &= \int_{\Sigma_{\tau_n}\cap[r_+,R_e]} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_{\tau_n}} + B\int_{\Sigma_{\tau_n}\cap[R_e,\infty)} \mathbb{J}^T_\mu[\psi]n^\mu_{\Sigma_{\tau_n}}\\ &\leq \frac{C}{\tau_n} + C\int_{\mathcal{H}^+(0,\tau_n)} \mathbb{J}^N_\mu[\psi]n^\mu_{\mathcal{H}^+} + C\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}.\end{aligned} \tag{3.6.11}$$

Adding an ϵ-multiple of the estimate of Proposition 3.6.4 to (3.3.6) and then applying (3.6.11), we have

$$\begin{aligned}&\int_0^{\tau_n}\int_{\Sigma_s\cap\{r\geq R_e\}} \left(r^{-1}\chi_\natural|\nabla\!\!\!/\psi|^2 + r^{-1-\delta}\chi_\natural(T\psi)^2 + r^{-1-\delta}(Z\psi)^2 + r^{-3-\delta}\psi^2\right)ds\\ &+\epsilon\int_0^{\tau_n}\int_{\Sigma_s\cap\{r\geq R_e\}} \left(r^{-1}\chi_\natural|\nabla\!\!\!/\psi|^2 + r^{-1-\delta}\chi_\natural(T\psi)^2 + r^{-1-\delta}(Z\psi)^2 + r^{-3-\delta}\psi^2\right)ds\\ &+(b-\epsilon)C\int_{\mathcal{H}^+(0,\tau_n)} \mathbb{J}^N_\mu[\psi]n^\mu_{\mathcal{H}^+} + C\int_{\mathcal{I}^+} \mathbb{J}^N_\mu[\psi]n^\mu_{\mathcal{I}^+}\\ &\leq B(\delta)\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0} + \frac{C}{\tau_n}.\end{aligned}$$

Taking ϵ small enough and letting $n\to\infty$ yields (3.3.7). Since we proved (3.3.6) in Proposition 3.6.3, this concludes the proof of the conditional Theorem 3.3.2.

3.7 The continuity argument

At this stage, we have proved Theorem 3.3.2 (and consequently Theorem 3.2.2) for solutions of (2.2.3) which are assumed to be sufficiently integrable in the sense of (3.3.1). We now remove this assumption by proving the following:

Proposition 3.7.1. *Let $M > 0$ and $a^2 + Q^2 \leq K_0^2 < M$. All solutions ψ to the wave equation* (2.2.3) *(arising from smooth, compactly supported initial data on Σ_0) are future integrable.*

We follow the strategy of [DRSR14] in first considering modes of fixed azimuthal frequency. Since Φ is a Killing field, it commutes with the D'Alembertian $\Box_g$. Thus for each azimuthal frequency $m \in \mathbb{Z}$, the projection P_m of ψ to its m^{th} azimuthal mode, $P_m\psi(t,r,\theta,\phi) = \tilde{\psi}(t,r,\theta)e^{im\phi}$, is well defined. Furthermore, $\Box_g(P_m\psi) = 0$.

3.7.1 The reduction to fixed azimuthal frequency

Lemma 3.7.1. *It suffices to prove Proposition 3.7.1 for solutions ψ to* (2.2.3) *supported on a single fixed azimuthal frequency $m \in \mathbb{Z}$.*

Proof. Let ψ solve (2.2.3). The fundamental theorem of calculus implies that

$$
\begin{aligned}
&B^{-1} \sup_{r\in[r_+,A]} \int_0^\infty \int_{\mathbb{S}^2} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!\!/^{i_1} T^{i_2}(Z)^{i_3}\psi|^2 \sin\theta \, dt \, d\theta \, d\phi \\
&\leq \int_{\mathcal{H}^+} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!\!/^{i_1} T^{i_2}(Z)^{i_3}\psi|^2 + \int_0^\infty \int_{\Sigma_s\cap[r_+,A]} \sum_{1\leq i_1+i_2+i_3\leq j+1} |\nabla\!\!\!\!/^{i_1} T^{i_2}(Z)^{i_3}\psi|^2 ds.
\end{aligned}
$$

Suppose we have established Proposition 3.7.1 for solutions supported on any fixed azimuthal frequency. We may then use the orthogonality of the azimuthal modes to expand $\psi = \sum_{m\in\mathbb{Z}} \psi_m$. Since each ψ_m is future-integrable, we have (ILED) and (3.2.1) for each ψ_m, verifying that (3.3.1) holds in the future of Σ_0. $\square$

Now it remains to prove the following:

Proposition 3.7.2. *Let $M > 0$, $a^2 + Q^2 \leq K_0^2 < M^2$, $m \in \mathbb{Z}$ and ψ be a solution of* (2.2.3) *that is supported only on the azimuthal frequency m. Then ψ satisfies* (3.3.1).

Our main tool in the proof of is a version of (ILED) for fixed mode azimuthal mode solutions, where we do **not** assume ψ is integrable a priori.

Lemma 3.7.2. *Under the hypotheses of Proposition 3.7.2, for every $\tau \geq 0$, $j \geq 1$ and $\delta > 0$,*

$$\int_0^\tau \int_{\Sigma_s} \left(r^{-1}\chi_\natural^m \left(|\nabla\!\!\!/\psi|^2 + r^{-\delta}|T\psi|^2\right) + r^{-1-\delta}|Z\psi|^2 + r^{-3-\delta}|\psi|^2 \right) ds + \int_{\mathcal{H}^+(0,\tau)} \mathbb{J}^N_\mu[\psi] n^\mu_{\mathcal{H}^+}$$
$$\leq B(\delta, m)\left(\int_{\Sigma_0} \mathbb{J}^N[\psi]\cdot n_{\Sigma_0} + \int_{\Sigma_\tau} \mathbb{J}^N[\psi]\cdot n_{\Sigma_\tau} \right) \tag{3.7.1}$$

and

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_s} r^{-1-\delta} \left(\sum_{1\leq i_1+i_2+i_3\leq j-1} \left(|\nabla\!\!\!/^{i_1} T^{i_2}(Z)^{i_3+1}\psi|^2 + |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi|^2 \right) \right.\\
&\qquad\qquad \left. + \chi_\natural^m \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2}(Z)^{i_3}\psi|^2 \right) ds \\
&+ \int_{\mathcal{H}^+(0,\tau)} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2}(Z)^{i_3}\psi|^2 \\
&\leq B(\delta, j, m)\left(\int_{\Sigma_0} \sum_{0\leq i\leq j-1} \mathbb{J}^N[N^i\psi]\cdot n_{\Sigma_0} + \int_{\Sigma_\tau} \sum_{0\leq i\leq j-1} \mathbb{J}^N[N^i\psi]\cdot n_{\Sigma_\tau} \right),
\end{aligned} \tag{3.7.2}$$

where $\chi_\natural^m = \left(1 - 1_{\{(1+\sqrt{2})M\leq r\leq R_\natural\}}\right)$.

Proof. For the first statement, we modify the cut-off γ of §3.3.1 and repeat the arguments of §3.6. That is, let $\gamma = 1$ identically between Σ_1 and $\Sigma_{\tau-1}$ and identically 0 to the past of Σ_0 and the future of Σ_τ. This allows us to remove the assumption that ψ is future integrable at the expense of picking up extra terms supported on Σ_τ on the right hand side of the estimates (3.7.1) and (3.7.2).

It is crucial that the degeneration of the estimate is encapsulated in $\chi_\natural^m$ rather than $\chi_\natural$. This is due to Lemma 3.4.5, which tells us that for fixed m and large Λ the trapped set is contained in $\{r \in [(1+\sqrt{2})\, M, \infty)\}$.

The second statement follows from the first in the same way that (3.2.2) follows from (3.2.1). □

Remark Recall from Lemma 3.4.5 that the degeneration due to trapping lies outside the ergoregion in this fixed azimuthal frequency case. This is extremely useful in the subsequent argument.

The following corollary will be our final reduction of the problem.

Corollary 3.7.3. *Under the hypotheses of Proposition 3.7.2, ψ is future-integrable if*

$$\sup_{\tau \geq 0} \int_{\Sigma_\tau} \sum_{1 \leq i_1 + i_2 + i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi|^2 < \infty \quad \forall\ j \geq 1. \tag{3.7.3}$$

Proof. As in the proof of Lemma 3.7.1, observe that

$$\begin{aligned}
&[B(j)]^{-1} \sup_{r \in [r_+, A]} \int_0^\infty \int_{\mathbb{S}^2} \sum_{1 \leq i_1 + i_2 + i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi|^2 \sin\theta \, dt \, d\theta \, d\phi \\
&\leq \int_{\mathcal{H}^+} \sum_{1 \leq i_1 + i_2 + i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi|^2 + \int_0^\infty \int_{\Sigma_s \cap [r_+, A]} \sum_{1 \leq i_1 + i_2 + i_3 \leq j+1} |\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi|^2 ds,
\end{aligned}$$

and apply (3.7.2) to the right hand side of this estimate. □

3.7.2 The setting and non-emptiness

We are now ready to run a continuity argument in the parameter Q to prove Proposition 3.7.2. Proposition 3.7.1 then follows immediately by Corollary 3.7.3.

We fix $M > 0$ and a such that $|a| < M$ and define for each $m \in \mathbb{Z}$, the set

$$\mathcal{Q}_{a,m} := \{Q^2 \in [0, M^2 - a^2) : (3.7.3) \text{ holds for } g = g_{a,Q,M}\}.$$

We will prove that $\mathcal{Q}_{a,m} = [0, M^2 - a^2)$ by showing that it is non-empty, open and closed. Proposition 3.7.2 then follows by Corollary 3.7.3.

We begin with non-emptiness.

Proposition 3.7.4. *For all $m \in \mathbb{Z}$ and a such that $|a| < M$, the set $\mathcal{Q}_{a,m}$ is non-empty.*

Proof. When $Q = 0$, the argument of [DRSR14, §11] shows that (3.7.3) holds. Thus $0 \in \mathcal{Q}_{a,m}$. □

Remark This appeal to the result of [DRSR14] is not necessary. We could just as well have run the continuity argument in both parameters a and Q, reproving the result of [DRSR14] in the process. In the interest of a clean and brief presentation, we just use the continuity of the metric in Q here.

3.7.3 Openness

In this section, we prove

Proposition 3.7.5. *For all $m \in \mathbb{Z}$ and a such that $|a| < M$, the set $\mathcal{Q}_{a,m}$ is open. That is, suppose $\mathring{Q} \in \mathcal{Q}_{a,m}$. Then there exists $\epsilon > 0$ such that*

$$\left|Q - \mathring{Q}\right| < \epsilon \Longrightarrow Q \in \mathcal{Q}_{a,m}.$$

The proof is in two parts. We first prove a derivative gaining (NEB)-type estimates in the spirit of §3.8.2 in §3.7.3. We then define a metric that interpolates between $g_{M,a,Q}$ and $g_{M,a,\mathring{Q}}$ and use the estimates of §3.7.1 and §3.7.3 to prove that if Q and $\mathring{Q}$ are close enough and (3.7.3) holds for $g_{M,a,Q}$, it also holds for $g_{M,a,\mathring{Q}}$.

Gaining derivatives

We begin by defining the following smooth, locally Killing, globally timelike vector field

Definition 3.7.3. *Let $a^2 + Q^2 \le K_0^2 < M^2$ and take $\epsilon_0 > 0$ as in Lemma 2.2.3. Let $\alpha(r)$ be a function such that $V := T + \alpha(r)\Phi$ is a smooth vector field, timelike in $\mathcal{M}$, satisfying*

$$\begin{aligned}
V &= T + \frac{a}{r_+^2 + a^2}\Phi, && \text{for } r \in [r_+, r_+ + \epsilon_0/2],\\
V &= T + \frac{a(r^2 + a^2 - \Delta)}{(r^2+a^2)^2}\Phi, && \text{for } r \in \left[r_+ + \epsilon_0, \frac{M\left(7+\sqrt{2}\right)}{4}\right],\\
V &= T, && \text{for } r \ge \frac{M\left(3+\sqrt{2}\right)}{2}.
\end{aligned}$$

Since $2M < \frac{M\left(3+\sqrt{2}\right)}{2} < M\left(1+\sqrt{2}\right)$, V is Killing in the region where trapping occurs for fixed azimuthal frequency m (see Lemma 3.4.5). Because of this, the error terms arising from the energy identity associated to V can be controlled by (3.7.1) and (3.7.2).

The following lemma is used as a converse to Lemma 3.7.2. However, we gain a derivative in the sense that the spacetime integral term on the right hand side of (3.7.4) is zeroth order.

Lemma 3.7.4. *Let $M > 0$, $a^2+Q^2 \le K_0^2 < M^2$, $m \in \mathbb{Z}$ and ψ be a solution of (2.2.3) that is supported only on the azimuthal frequency m. Then there exists a constant $B = B(m)$ such that for all $\tau \ge 0$,*

$$\begin{aligned}
\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau} &\le B(m)\left(\int_0^\tau \int_{\Sigma_s \cap \left\{r \le \frac{M\left(3+\sqrt{2}\right)}{2}\right\}} |\Phi\psi|^2 \, ds + \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}\right)\\
&\le B(m)\left(\int_0^\tau \int_{\Sigma_s \cap \left\{r \le \frac{M\left(3+\sqrt{2}\right)}{2}\right\}} |\psi|^2 \, ds + \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}\right). \qquad (3.7.4)
\end{aligned}$$

Proof. Applying the energy identity associated to the vector field V, we have

$$\int_{\Sigma_\tau} \mathbb{J}^V_\mu[\psi] n^\mu_{\Sigma_\tau} \leq B \int_0^\tau \int_{\Sigma_s} \left|\mathbb{K}^V[\psi]\right| \, ds + \int_{\Sigma_0} \mathbb{J}^V_\mu[\psi] n^\mu_{\Sigma_0}.$$

It remains to control the spacetime integral term. We observe that $\mathbb{K}^V[\psi] = 0$ outside $\mathrm{supp}(\frac{d\alpha}{dr})$ and

$$\left|\mathbb{K}^V[\psi]\right| \leq B \left(\epsilon|\partial_r\psi|^2 + \epsilon^{-1}|\Phi\psi|^2\right),$$

and apply (3.7.1) to the first term:

$$\epsilon \int_0^\tau \int_{\Sigma_s \cap \mathrm{supp}(\frac{d\alpha}{dr})} |\partial_r\psi|^2 \, ds \leq B(m)\epsilon \left(\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau} + \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}\right).$$

Adding this estimate to the energy identity above, we have

$$\begin{aligned} \int_{\Sigma_\tau} \mathbb{J}^V_\mu[\psi] n^\mu_{\Sigma_\tau} \leq B(m) &\left(\epsilon^{-1} \int_0^\tau \int_{\Sigma_s \cap \left\{r \leq \frac{M(3+\sqrt{2})}{2}\right\}} |\Phi\psi|^2 \, ds + \epsilon \int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau}\right) \\ &+ B(m) \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}. \end{aligned}$$

The proof is completed by applying the argument presented in [DRSR14, §11.2]. □

The proof above does not use the fact that the ergoregion and trapping region are disjoint. Rather, the non-degeneracy of the $(\partial_r\psi)^2$ term in (3.7.1) is used. Therefore, we could prove the first line of (3.7.4) without the restriction to fixed m.

For fixed ψ supported on a fixed, azimuthal frequency m however, we have $|\Phi\psi| = |m\psi|$. Since m is fixed, the presence of the ergoregion is only a low-frequency obstruction to proving (NEB).

The proof of Lemma 3.7.4 is similar to Proposition 3.8.3 in that we obtain a nondegenerate boundedness estimate without the use of a fully nondegenerate integrated local energy decay estimate.

By combining (3.7.2) and (3.7.4), we obtain the following higher order version of (3.7.4).

Lemma 3.7.5. *Let $M > 0$, $a^2 + Q^2 \leq K_0^2 < M^2$, $m \in \mathbb{Z}$ and ψ be a solution of* (2.2.3) *that is supported only on the azimuthal frequency m. Then, for every $j \geq 1$, and all $\tau \geq 0$,*

$$\int_{\Sigma_\tau} \sum_{1 \leq i_1+i_2+i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi|^2$$

$$\leq B(j,m) \left(\int_0^\tau \int_{\Sigma_s \cap \left\{ r \leq \frac{M(3+\sqrt{2})}{2} \right\}} |\psi|^2 \, ds + \int_{\Sigma_0} \sum_{1 \leq i_1+i_2+i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} (Z)^{i_3} \psi|^2 \right). \quad (3.7.5)$$

Proof. The additional Q-terms that arise in passing from the Kerr to the Kerr–Newman case are harmless in the derivation of the higher order result, so the argument of [DRSR14, §11.2] may be applied directly. □

Combining (3.7.2) and (3.7.5), we obtain the following useful corollary.

Corollary 3.7.6. *Let $M > 0$, $a^2 + Q^2 \leq K_0^2 < M^2$, $m \in \mathbb{Z}$ and ψ be a solution of* (2.2.3) *that is supported only on the azimuthal frequency m. Then, for every $\delta > 0$, $j \geq 1$, and all $\tau \geq 0$,*

$$\sup_{\tau' \leq \tau} \int_{\Sigma_{\tau'}} \sum_{1 \leq i_1+i_2+i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi|^2$$

$$+ \int_0^\tau \int_{\Sigma_s} r^{-1-\delta} \left(r^{-2} |\psi|^2 + 1_{[r_+, (1+\sqrt{2})M]} \sum_{1 \leq i_1+i_2+i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi|^2 \right) ds$$

$$+ \int_0^\tau \int_{\Sigma_s} r^{-1-\delta} \left(\sum_{1 \leq i_1+i_2+i_3 \leq j-1} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi|^2 + \sum_{1 \leq i_1+i_2+i_3 \leq j-1} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3+1} \psi|^2 \right) ds$$

$$\leq B(\delta, j, m) \left(\int_0^\tau \int_{\Sigma_s \cap \left\{ r \leq \frac{M(3+\sqrt{2})}{2} \right\}} |\psi|^2 \, ds + \int_{\Sigma_0} \sum_{1 \leq i_1+i_2+i_3 \leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi|^2 \right). \quad (3.7.6)$$

The interpolating metric

We now prove Proposition 3.7.5.

Proof of Proposition 3.7.5. Consider fixed $M > 0$, fixed $a^2 \leq a_0^2 < M^2$ and fixed $m \in \mathbb{Z}$. Let $\mathring{Q} \in \mathcal{Q}_{a,m}$. Then choose Q_0 such that $\mathring{Q}^2 < Q_0^2 < M^2 - a^2$. Our aim is to find an $\epsilon > 0$ satisfying $\mathring{Q}^2 + \epsilon^2 \leq Q_0^2$ such that

$$\left| Q - \mathring{Q} \right| < \epsilon \Longrightarrow Q \in \mathcal{Q}_{a,m}.$$

Let ψ be a solution of (2.2.3) on $g_{M,a,Q}$ that is supported only on the azimuthal frequency m and take Q such that $\left| Q - \mathring{Q} \right| < \epsilon$ for an ϵ to be determined.

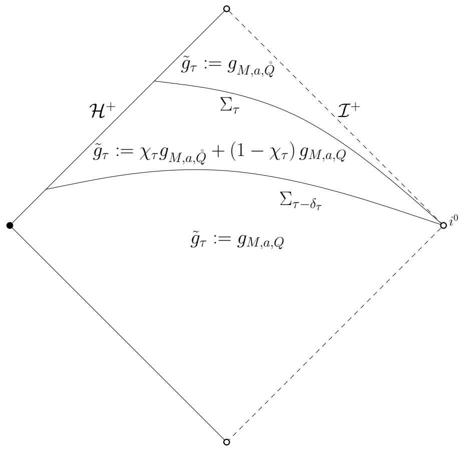

Figure 3.7.1: The interpolating metric.

Let $\tilde{\psi}$ be a solution of (2.2.3) on $g_{M,a,\mathring{Q}}$, also supported only the single azimuthal frequency m. Since $\mathring{Q} \in \mathcal{Q}_{a,m}$, Corollary 3.7.3 implies that $\tilde{\psi}$ is future-integrable. We will exploit this by introducing a metric $\tilde{g}_\tau$ which interpolates between $g_{M,a\mathring{Q}}$ and $g_{M,a,Q}$:

Definition 3.7.6. *Set $\tau \geq 1$. Let*

$$\chi_\tau = \begin{cases} 1 & \text{in the future of } \Sigma_\tau \\ 0 & \text{in the past of } \Sigma_{\tau-\delta_\tau}, \\ \text{smooth} & \text{between } \Sigma_{\tau-\delta_\tau} \text{ and } \Sigma_\tau \end{cases}$$

for sufficiently small $\delta_\tau > 0$. Now define the interpolating metric $\tilde{g}_\tau$ by

$$\tilde{g}_\tau := \chi_\tau g_{M,a,\mathring{Q}} + (1-\chi_\tau)\, g_{M,a,Q}. \tag{3.7.7}$$

See Figure 3.7.1.

If ϵ is small enough, $\tilde{g}_\tau$ is a Lorentzian metric on $\mathcal{M}$.

With our interpolating metric defined, we define the solution to its wave equation.

Definition 3.7.7. *Let ψ be the solution of $\Box_{g_{M,a,Q}}\psi = 0$ defined above. Let $\tilde{\psi}_\tau$ be the interpolating solution of $\Box_{\tilde{g}_\tau}\tilde{\psi}_\tau = 0$ with the same initial data as ψ on Σ_0.*

This is well defined since Σ_τ is a past Cauchy hypersurface for the future of Σ_τ with respect to $\tilde{g}_\tau$ (which is identically equal to $g_{M,a,\mathring{Q}}$ in that region).

Now $\tilde{\psi}_\tau = \psi$ in the past of $\Sigma_{\tau-\delta_\tau}$ and $\Box_{g_{M,a,\mathring{Q}}}\tilde{\psi}_\tau = 0$ in the future of Σ_τ.

Since Φ is a Killing vector field for $g_{M,a,Q}$ and $g_{M,a,\mathring{Q}}$ and χ_τ does not depend on Φ, it is clear that Φ is Killing for the metric $\tilde{g}_\tau$. Therefore, the interpolating solution $\tilde{\psi}_\tau$ and the as the original solution ψ will be supported on the same azimuthal frequency m.

The assumption $\mathring{Q} \in \mathcal{Q}_{a,m}$ allows us to use Corollary 3.7.3 to conclude that $\tilde{\psi}_\tau$ is future-integrable with respect to $\mathring{Q}$.

Note that

$$\Box_{g_{M,a,\mathring{Q}}} \tilde{\psi}_\tau = \left(\Box_{g_{M,a,\mathring{Q}}} - \Box_{\tilde{g}_\tau}\right) \tilde{\psi}_\tau.$$

A computation then shows that

$$r^{1+\delta} \left| \left(\Box_{g_{M,a,\mathring{Q}}} - \Box_{\tilde{g}_\tau}\right) \tilde{\psi}_\tau \right|^2 \leq B\left(\delta_\tau^{-1}\right) \left|Q - \mathring{Q}\right|^2 r^{-2} \sum_{1 \leq i_1+i_2+i_3 \leq 2} \left| \nabla\!\!\!\!/^{i_1} T^{i_2} Z^{i_3} \tilde{\psi}_\tau \right|^2. \quad (3.7.8)$$

This structure is essential to proving the desired estimate. With (3.7.8) established, the argument follows the same logic as in [DRSR14, §11.2].

In what follows, metric defined quantities refer to $g_{M,a,\mathring{Q}}$.

The error term (3.7.8) is supported only in the past of Σ_τ, so Proposition 3.3.1 followed by (ILED) for $g_{M,a,\mathring{Q}}$ imply

$$\begin{aligned}
&\int_0^\tau \int_{\Sigma_s \cap \left\{r \leq \frac{M(3+\sqrt{2})}{2}\right\}} \left(\mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_s} + |\psi|^2 \right) ds \\
&\leq \int_0^{\tau-\delta_\tau} \int_{\Sigma_s \cap \left\{r \leq M\left(1+\sqrt{2}\right)\right\}} \left(\mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_s} + |\psi|^2 \right) ds \\
&\leq B(\delta_\tau, m) \left|Q - \mathring{Q}\right|^2 \int_0^\tau \int_{\Sigma_s} \sum_{1 \leq i_1+i_2+i_3 \leq 2} r^{-2} \left| \nabla\!\!\!\!/^{i_1} T^{i_2} Z^{i_3} \tilde{\psi}_\tau \right|^2 ds \\
&+ B\left(\delta_\tau, m\right) \left|Q - \mathring{Q}\right|^2 \int_\tau^\infty \int_{\Sigma_s \cap \left[r_+, \left(1+\sqrt{2}\right)M\right]} \left[\left| T \tilde{\psi}_\tau \right|^2 + |\psi|^2 \right] ds \\
&+ B(m) \int_{\Sigma_0} \left[\mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0} + |\psi|^2 \right], \qquad (3.7.9)
\end{aligned}$$

for δ_τ sufficiently small.

Again by Proposition 3.3.1 (applied to second order derivatives of $\tilde{\psi}_\tau$ and those of ψ respectively),

$$\begin{aligned}
&\int_{\tau-\delta_\tau}^\tau \int_{\Sigma_s} \sum_{1 \leq i_1+i_2+i_3 \leq 2} r^{-2} \left(\left| \nabla\!\!\!\!/^{i_1} T^{i_2} Z^{i_3} \tilde{\psi}_\tau \right|^2 + \left| \nabla\!\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi \right|^2 \right) ds \\
&\leq B \int_{\Sigma_{\tau-\delta_\tau}} \sum_{1 \leq i_1+i_2+i_3 \leq 2} \left| \nabla\!\!\!\!/^{i_1} T^{i_2} Z^{i_3} \psi \right|^2. \qquad (3.7.10)
\end{aligned}$$

Since $\tilde{\psi}_\tau$ is future integrable, we can apply (3.2.2), and Proposition 3.3.1 again, to arrive at

$$\int_{\tau}^{\infty}\int_{\Sigma_s\cap[r_+,(1+\sqrt{2})M]}\left[\left|T\tilde{\psi}_\tau\right|^2+|\psi|^2\right]\leq B\int_{\Sigma_{\tau-\delta_\tau}}\sum_{1\leq i_1+i_2+i_3\leq 2}\left|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi\right|^2. \quad (3.7.11)$$

Combining (3.7.9), (3.7.10) and (3.7.11) gives

$$\begin{aligned}
&\int_0^{\tau}\int_{\Sigma_s\cap\left\{r\leq\frac{M(3+\sqrt{2})}{2}\right\}}\left(\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_s}+|\psi|^2\right)ds\\
\leq& B(m)\left|Q-\mathring{Q}\right|^2\int_{\Sigma_{\tau-\delta_\tau}}\sum_{1\leq i_1+i_2+i_3\leq 2}\left|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\tilde{\psi}_\tau\right|^2\\
&+B(m)\left|Q-\mathring{Q}\right|^2\int_0^{\tau}\int_{\Sigma_s}\sum_{1\leq i_1+i_2+i_3\leq 2}r^{-2}\left|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi\right|^2ds \qquad (3.7.12)\\
&+B(m)\int_{\Sigma_0}\left[\mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}+|\psi|^2\right].
\end{aligned}$$

Applying Corollary 3.7.6 followed by (3.7.12):

$$\begin{aligned}
&\sup_{\tau'\leq\tau}\int_{\Sigma_{\tau'}}\sum_{1\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2\\
&+\int_0^{\tau}\int_{\Sigma_s}r^{-1-\delta}\left(r^{-2}|\psi|^2+1_{[r_+,(1+\sqrt{2})M]}\sum_{1\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2\right)ds\\
&+\int_0^{\tau}\int_{\Sigma_s}r^{-1-\delta}\left(\sum_{1\leq i_1+i_2+i_3\leq j-1}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2+\sum_{1\leq i_1+i_2+i_3\leq j-1}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3+1}\psi|^2\right)ds\\
\leq& B(\delta,j,m)\left(\int_0^{\tau}\int_{\Sigma_s\cap\left\{r\leq\frac{M(3+\sqrt{2})}{2}\right\}}|\psi|^2\,ds+\int_{\Sigma_0}\sum_{1\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2\right)\\
\leq& B(\delta,j,m)\left|Q-\mathring{Q}\right|^2\int_0^{\tau}\int_{\Sigma_s}\sum_{1\leq i_1+i_2+i_3\leq 2}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2ds\\
&+B(\delta,j,m)\left|Q-\mathring{Q}\right|^2\int_0^{\tau}\int_{\Sigma_s}r^{-2}\sum_{1\leq i_1+i_2+i_3\leq 2}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2\,ds\\
&+B(\delta,j,m)\int_{\Sigma_0}\sum_{0\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2.
\end{aligned}$$

Taking $j\geq 3$ and letting ϵ be small enough that we can absorb the $\left|Q-\mathring{Q}\right|^2$ term on the left hand side. We conclude that

$$\sup_{\tau'\leq\tau}\int_{\Sigma_{\tau'}}\sum_{1\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2\leq B(j,m)\int_{\Sigma_0}\sum_{0\leq i_1+i_2+i_3\leq j}|\nabla\!\!\!/^{i_1}T^{i_2}Z^{i_3}\psi|^2<\infty.$$

Since τ was chosen arbitrarily and the right hand side of this estimate is independent of τ, this is the sufficient condition for integrability of Corollary 3.7.3. We have therefore shown that there exists an ϵ small enough that $\left|Q - \mathring{Q}\right| < \epsilon \Longrightarrow Q \in \mathcal{Q}_{a,m}$. $\square$

3.7.4 Closedness

To close the continuity argument and complete the proof of Proposition 3.7.2, it remains to prove

Proposition 3.7.7. *The set $\mathcal{Q}_{a,m}$ is closed in $[0, M^2 - a^2)$. That is, if the sequence $\{Q_k\}_{k=1}^{\infty} \subset \mathcal{Q}_{a,m}$ and $Q_k^2 < M^2 - a^2$, then $Q \in \mathcal{Q}_{a,m}$.*

Proof. As in the statement of Proposition 3.7.2, let ψ be a solution the wave equation $\Box_{g_{M,a,Q}}\psi = 0$ supported on a fixed azimuthal frequency m. Set $Q_0 < M$ such that $Q^2 < Q_0^2$. Without loss of generality, we assume that $Q_k^2 \leq Q_0^2$ for all k.

Now define the sequence of functions ψ_k to be solutions of $\Box_{g_{M,a,Q_k}}\psi_k = 0$ with the same initial data as ψ. Then by Lemma 2.2.1,

$$\int_{\Sigma_\tau} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi|^2 = \lim_{k\to\infty} \int_{\Sigma_\tau} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi_k|^2 \tag{3.7.13}$$

for every $\tau \geq 0$, $j \geq 1$. Since each $Q_k \in \mathcal{Q}_{a,m}$, each ψ_k is future integrable. We may then apply (3.2.3) to each ψ_k: for every $j \geq 1$,

$$\sup_{\tau\geq 0} \int_{\Sigma_\tau} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi_k|^2 \leq B(j,m) \int_{\Sigma_0} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi_k|^2. \tag{3.7.14}$$

Combining (3.7.13) and (3.7.14), we conclude that

$$\begin{aligned}\sup_{\tau\geq 0} \int_{\Sigma_\tau} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi|^2 &\leq B(j,m) \lim_{k\to\infty} \int_{\Sigma_0} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi_k|^2 \\ &= B(j,m) \int_{\Sigma_0} \sum_{1\leq i_1+i_2+i_3\leq j} |\nabla\!\!\!/^{i_1} T^{i_2} Z^{i_3}\psi|^2,\end{aligned}$$

which is (3.7.3), so $Q \in \mathcal{Q}_{a,m}$. $\square$

3.7.5 Proof of (ILED)

We have now proved Proposition 3.7.2, the integrability result for solutions supported on fixed azimuthal frequency. By the reduction given by Lemma 3.7.1, Proposition 3.7.1

follows directly. We have therefore shown that any solution of (2.2.3) on a subextremal Kerr–Newman exterior spacetime is future integrable. This allows us to appeal to the conditional Theorem 3.3.2 for the full range of solutions. From this we conclude that (ILED) in Theorem 3.2.1 holds unconditionally.

3.8 Proof of (NEB)

At this point we have proved that for $a^2 + Q^2 < M^2$, every solution ψ of (2.2.3) is future-integrable and moreover satisfies the integrated decay statements (ILED) and (3.2.1). We now prove (NEB). Naively, one may apply the energy identity for N,

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_\tau} \leq \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0} - \int_0^\tau \int_{\Sigma_t} \mathbb{K}^N[\psi]\ dt^* \tag{3.8.1}$$

and attempt to control the last term on the right hand side by (3.2.1). In the case that $a^2+Q^2 \ll M^2$, this approach works. This is because one has a small parameter to exploit, which allows the vector field N to be chosen in such a way that $\mathbb{K}^N[\psi]$ is not supported in the physical space projection of the trapped set. See [DR11a] for the details.

In general, this approach fails due to the degeneracies of (3.2.1). For the full subextremal Kerr–Newman case, we turn to a more sophisticated approach which employs phase space localisation of ψ and specific features of the Kerr–Newman geometry.

We begin by recalling from Lemmas 3.4.1 and 3.5.2 that there exists a constant $R_\natural$ such that the degeneration of estimate (3.5.3) due to trapping only occurs in $\mathcal{R}(0,\tau) \cap \{r_e < r < R_\natural\}$ where

$$r_e < R_\natural \ll R_e.$$

Theorem 3.2.1 ensures that we can fix R_e large enough in (ILED) to satisfy the inequality above. Cover $\mathcal{M}$ by the sets

$$\begin{aligned} \tilde{A}_H &= \mathcal{M} \cap \{r_+ \leq r < r_e - \epsilon\}, \\ \tilde{A}_{trap} &= \mathcal{M} \cap \{r_e - 2\epsilon < r < R_\natural + 2\epsilon\} \\ \text{and } \tilde{A}_R &= \mathcal{M} \cap \{r > R_\natural + \epsilon\}, \end{aligned}$$

where the presence of $\epsilon > 0$ ensures that the trapping region is strictly contained in $\tilde{A}_{trap}$. Let us take a smooth partition of unity $\{\chi_H(r), \chi_{trap}(r), \chi_R(r)\}$ subordinate to the cover $\left\{\tilde{A}_H, \tilde{A}_{trap}, \tilde{A}_R\right\}$ with

$$\text{supp } \chi_{\{H,trap,R\}}(r) = \tilde{A}_{\{H,trap,R\}}.$$

Now for a solution of (2.2.3) we can write

$$\psi = \psi_H + \psi_{trap} + \psi_R, \tag{3.8.2}$$

where

$$\psi_{\{H,trap,R\}} = \chi_{\{H,trap,R\}} \cdot \psi.$$

Clearly each $\psi_{\{H,trap,R\}}$ has the same smoothness and integrability properties as ψ. Furthermore,

$$\Box_g \psi_{\{H,trap,R\}} = G_{\{H,trap,R\}}, \tag{3.8.3}$$

where

$$G_{\{H,trap,R\}} = (\Box(\chi_{\{H,trap,R\}}))\psi + 2\nabla^\mu(\chi_{\{H,trap,R\}})\nabla_\mu\psi.$$

Outside the trapping region $\tilde{A}_{trap}$, the (NEB) statement may be extracted from (ILED) in a direct manner. We will consider this first.

3.8.1 (NEB) outside the trapping region

(NEB) near the horizon

Applying the energy identity for N to ψ_H, we have

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi_H] n^\mu_{\Sigma_\tau} \leq \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi_H] n^\mu_{\Sigma_0} - \int_0^\tau \int_{\Sigma_t \cap \{r_+ \leq r \leq r_e - \epsilon\}} (\mathbb{K}^N[\psi_H] + \mathcal{E}^N[\psi_H]) \; dt^*.$$

Since χ_H is smooth with compactly supported derivatives,

$$\begin{aligned} \left|\mathcal{E}^N[\psi_H]\right| &= |[N(\chi_H)\psi + (\chi_H)(N\psi)][\Box(\chi_H)\psi + 2\nabla^\mu(\chi_H)(\nabla_\mu\psi)]| \\ &\leq C\psi^2 + C|\partial\psi|^2. \end{aligned}$$

So

$$\int_0^\tau \int_{\Sigma_t \cap \{r_+ \leq r \leq r_e - \epsilon\}} \left|\mathbb{K}^N[\psi_H]\right| + \left|\mathcal{E}^N[\psi_H]\right| \; dt^* \leq C \int_0^\tau \int_{\Sigma_t \cap \{r_+ \leq r \leq r_e - \epsilon\}} \psi^2 + |\partial\psi|^2 \; dt^*.$$

The estimate (3.6.8) does not degenerate in $\tilde{A}_H$, so we may apply it to the spacetime integral above to obtain

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi_H] n^\mu_{\Sigma_\tau} \leq \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi_H] n^\mu_{\Sigma_0} + C \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}.$$

The zeroth order terms in $\mathbb{J}^N_\mu[\psi_H]n^\mu_{\Sigma_0}$ can be controlled using the Hardy inequality (2.2.8) in r (we exploit the smoothness of the cut-offs and the compact support of ψ_H). Hence

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi_H] \leq (C_H + C)\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0}. \tag{3.8.4}$$

(NEB) for large r

Here we apply the energy identity for T to ψ_R,

$$\int_{\Sigma_\tau} \mathbb{J}^T_\mu[\psi_R]n^\mu_{\Sigma_\tau} \;\leq\; \int_{\Sigma_0} \mathbb{J}^T_\mu[\psi_R]n^\mu_{\Sigma_0} - \int_0^\tau \int_{\Sigma_t \cap \left\{r \geq R_\natural + \epsilon\right\}} \mathcal{E}^T[\psi_R]\; dt^*.$$

Note that

$$\begin{aligned}
\int_{\Sigma_0} \mathbb{J}^T_\mu[\psi_R]n^\mu_{\Sigma_0} &\leq C\int_{\Sigma_0} |\nabla(\chi_R\psi)|^2 \\
&\leq C\int_{\Sigma_0} |\chi_R(\nabla\psi) + (\partial_r\chi_R)(\psi)|^2 \\
&\leq C\int_{\Sigma_0} |(\partial\psi)|^2 + C\int_{\Sigma_0\cap\left\{R_\natural+\epsilon\leq r\leq R_\natural+2\epsilon\right\}} (\partial_r\chi_R)^2(\psi)^2 \\
&\leq C(R_\natural + 2\epsilon)^2 \int_0^\tau \int_{\Sigma_t\cap\left\{R_\natural+\epsilon\leq r\leq R_\natural+2\epsilon\right\}} r^{-2}\psi^2 + |\partial\psi|^2\; dt^* \\
&\leq C\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0},
\end{aligned}$$

where we have used the Hardy inequality (2.2.8) in r. The spacetime integrand is

$$\begin{aligned}
\mathcal{E}^T[\psi_R] &= [T(\chi_R)\psi + (\chi_R)(T\psi)][\Box(\chi_R)\psi + 2\nabla^\mu(\chi_R)(\nabla_\mu\psi)] \\
&= (\chi_R)(T\psi)[\Box(\chi_R)\psi + 2\nabla^\mu(\chi_R)(\nabla_\mu\psi)] \\
&= \left(\frac{\partial_r(\Delta\partial_r\chi_R)}{\rho^2\sin\theta}\right)(\psi)(\chi_R)(T\psi) + 2(g^{rr})(\chi_R)(\partial_r\chi_R)(\partial_r\psi)(T\psi)
\end{aligned}$$

Observe that each term in $\mathcal{E}^T[\psi_R]$ contains factors of $(\partial_r\chi_R)$ or $(\partial_r^2\chi_R)$ and is consequently supported in $\{R_\natural + \epsilon \leq r \leq R_\natural + 2\epsilon\}$. Applying Cauchy Schwarz and using the boundedness of the metric components and cut-offs,

$$\begin{aligned}
\int_0^\tau \int_{\Sigma_t\cap\left\{r\geq R_\natural+\epsilon\right\}} \left|\mathcal{E}^T[\psi_R]\right| dt^* &= \int_0^\tau \int_{\Sigma_t\cap\left\{R_\natural+\epsilon\leq r\leq R_\natural+2\epsilon\right\}} \left|\mathcal{E}^T[\psi_R]\right| dt^* \\
&\leq C\int_0^\tau \int_{\Sigma_t\cap\left\{R_\natural+\epsilon\leq r\leq R_\natural+2\epsilon\right\}} \psi^2 + |\partial\psi|^2\; dt^* \\
&\leq C\int_{\Sigma_0} \mathbb{J}^N_\mu[\psi]n^\mu_{\Sigma_0},
\end{aligned}$$

where we have used (3.6.8) in the last inequality (note that this estimate does not degenerate in the region under consideration). We conclude that

$$\int_{\Sigma_\tau} \mathbb{J}^N_\mu[\psi_R] \leq C_R \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}. \tag{3.8.5}$$

3.8.2 The set up for the proof of (NEB) in the trapping region

Recall the decomposition (3.8.2),

$$\psi(t^*, r, \theta, \phi) = (\chi_H(r) + \chi_R(r) + \chi_{trap}(r))\, \psi(t^*, r, \theta, \phi).$$

We have proved (NEB) for $(\chi_H + \chi_R)\psi$ in (3.8.4) and (3.8.5). The proof of (NEB) for $\chi_{trap}\psi$ requires a more sophisticated approach which employs features of the Kerr–Newman geometry and localisation of ψ in phase space.

The idea is to project to finitely many *wave packets* for which the degeneration of (3.6.8) is contained in a particular bounded $r-$interval. Then a bespoke energy current can be constructed for each wave packet so that, upon application of the associated energy identity, we obtain (NEB) for each wave packet. It then remains to sum over the wave packets to obtain (NEB) for the full solution of (2.2.3).

We first describe the relevant geometric features. The localisation depends on these features and follows in §3.8.3.

It will be convenient to extend the solution ψ of (2.2.3) from the future of Σ_0 to the entire domain of outer communications.

Extending to the past

The initial data in (2.2.3) only determine the solution ψ in the future of Σ_0. We extend the solution ψ to the entire domain of outer communications as follows:

Consider the maximal globally hyperbolic extension of the Kerr–Newman manifold $\mathcal{M}_e$ (see for example [Car73]). Denote the extension of Σ_0 as a spacelike hypersurface to $\mathcal{M}_e$ by Σ_0^e. Now Σ_0^e is a Cauchy hypersurface for $\mathcal{M}_e$.

We can extend the initial data on Σ_0 to Σ_0^e as C^1 functions in such a way that

$$\int_{\Sigma_0^e} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0^e} \leq C \int_{\Sigma_0} \mathbb{J}^N_\mu[\psi] n^\mu_{\Sigma_0}. \tag{3.8.6}$$

Since Σ_0^e is a Cauchy hypersurface for $\mathcal{M}_e$, we can solve the initial value problem both forwards and backwards in such a way that the solution is $C^2(\mathcal{M})$ and agrees with the solution ψ of (2.2.3) in $\mathcal{M} \cap J^+(\Sigma_0)$.

From here on we consider the extended solution, which we continue to denote by ψ.

Let $\tilde{\Sigma}_0^e$ be the image of Σ_0^e under the map $t \mapsto -t$, where t is the Boyer–Lindquist coordinate of §2.1.1. Then applying the energy identity for $n_{\Sigma_0^e}$ between Σ_0^e and $\tilde{\Sigma}_0^e$, the proof of Proposition 3.3.1 and the analogue of (3.8.6) for Σ_0^e we have

$$\int_{\tilde{\Sigma}_0} \mathbb{J}_\mu^N[\psi] n^\mu_{\tilde{\Sigma}_0} \leq C \int_{\Sigma_0} \mathbb{J}_\mu^N[\psi] n^\mu_{\Sigma_0}. \tag{3.8.7}$$

We now extend (ILED) to the past of Σ_0.

Corollary 3.8.1. *Let ψ solve* (2.2.3)*. For any $r_+ < r_0 < R_0 \leq R_e$ there exists a positive constant C_{r_0,R_0} such that*

$$\int_{-\infty}^{\infty} \int_{\Sigma_\tau \cap \{r_0 \leq r \leq R_0\}} \left(\chi_\natural |\partial\psi|^2 + |\psi|^2 \right) dt^* \leq C_{r_0,R_0} \int_{\Sigma_0} |\partial\psi|^2, \tag{3.8.8}$$

Here $\chi_\natural$ is a cut-off function that vanishes in a neighbourhood of the physical space projection of the trapped set, see §3.5.3.

Proof. Write

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{\Sigma_s \cap \{r_0 \leq r \leq R_0\}} \left(\chi_\natural |\partial\psi|^2 + |\psi|^2 \right) ds \\
&= \left(\int_{J^+(\Sigma_0) \cap \{r_0 \leq r \leq R_0\}} + \int_{\left\{J^-(\Sigma_0) \cap J^+(\tilde{\Sigma}_0)\right\} \cap \{r_0 \leq r \leq R_0\}} \right) \left(\chi_\natural |\partial\psi|^2 + |\psi|^2 \right) \\
&\quad + \int_{J^-(\tilde{\Sigma}_0) \cap \{r_0 \leq r \leq R_0\}} \left(\chi_\natural |\partial\psi|^2 + |\psi|^2 \right).
\end{aligned}$$

For the first integral, we simply apply (ILED). For the last integral, we note that mapping $(t, a) \mapsto (-t, -a)$ is an isometry, so (ILED) holds with Σ_0 replaced by $\tilde{\Sigma}_0$ and the integrals $(0, \tau)$ replaced by integrals over $(-\tau, 0)$. The remaining term is controlled Proposition 3.3.1 and (3.8.7). □

Locally Killing, globally timelike vector fields

Let us first recall the following property of subextremal Kerr–Newman spacetimes: For any $\tilde{r} \geq r_+$, there exists a constant $c = c(\tilde{r})$ such that the smooth vector field

$$Q_c = T + c\Phi$$

is timelike and Killing at $\tilde{r}$. By continuity of the metric, there exists an open set of the form $A = \{r_a < r < R_A\}$ containing this $\tilde{r}$ for which Q_c remains timelike and Killing in

A. We extend Q_c to a globally defined smooth vector field by defining

$$Q_A = T + b_A(r)\Phi$$

where $b_A(r)$ is a smooth function chosen such that Q_A is smooth and globally timelike in $R(0,\tau)$ with the further property that $b_A(r) = c$ in A.

Covering the trapping region

Our goal is to overcome the degeneration of (3.6.8) in a neighbourhood of the physical space projection of the trapped set $\left\{r \ : \ r \in \bigcap_{L=1}^{\infty} \bigcup_{\ell \geq L} r_{m\ell}^{(a\omega)}\right\}$. This degeneration arose from the fact that the estimate (3.5.3) must degenerate on $\left\{r = r_{m\ell}^{(a\omega)}\right\}_{(\omega,m,\ell)}$ for each trapped mode. Recall from Lemmas 3.4.1 and 3.5.2 that these $r_{m\ell}^{(a\omega)}$ must lie in $\mathcal{R}(0,\tau) \cap \{r_e < r < R_\natural\} \subset \tilde{A}_{trap}$.

By compactness and the construction in §3.8.2, $\tilde{A}_{trap}$ can be covered by finitely many open sets, say $\{A_n\}_{n=1}^{\tilde{N}}$, for which the smooth vector field

$$Q^{A_n} = T + c_n\Phi$$

is timelike and Killing in A_n. We now use Q^{A_n} to define the vector field

$$Q_n = T + b_n(r)\Phi$$

where $b_n(r)$ is a smooth function chosen such that Q_n is smooth and globally timelike in $R(0,\tau)$ with the further property that

$$Q_n = \begin{cases} N & \text{for } \ R(0,\tau) \cap \{r < r_e - 3\epsilon\}\,, \\ Q^{A_n} & \text{in } \ A_n, \\ T & \text{for } \ R(0,\tau) \cap \{r > R_\natural + 3\epsilon\}\,. \end{cases} \tag{3.8.9}$$

We emphasise that Q_n is smooth and globally timelike and in particular, a local Killing vector field in A_n.

For $n = 0$ we just let $Q_0 = N$.

A refinement

Ultimately we want to apply energy currents $\mathbb{J}^{Q_n}$ to appropriately localised solutions of (3.3.4), which we will call *wave packets* (the precise definition is given later in (3.8.10)). We want to construct each wave packet in such a way that (3.5.11) applied to the n^{th} wave packet is nondegenerate in $\mathcal{R}(0,\tau) \setminus A_n$.

To ensure this, we will need a refinement $\left\{\tilde{A}_n\right\}_{n=1}^{\tilde{N}}$ of $\{A_n\}_{n=1}^{\tilde{N}}$ such that the degeneration of (3.5.11) applied to the n^{th} wave packet is strictly contained in $\tilde{A}_n \subset A_n$. Denote each A_n by

$$A_n = (r_{A_n}, R_{A_n}) \quad \text{where } r_{A_n} < r_{A_{n+1}} < R_{A_n} < R_{A_{n+1}}.$$

Observe that the intersections of adjacent A_n are nonempty open sets of the form

$$A_{n-1} \cap A_n = (r_{A_n}, R_{A_{n-1}}).$$

Hence we can define

$$\tilde{A}_n = (r_{A_n} + \epsilon_n, R_{A_n} - \epsilon_n)$$

where each ϵ_n is chosen small enough that the intersections of adjacent $\tilde{A}_n$ are nonempty open sets.

3.8.3 Construction of wave packets

We now construct the wave packets by localising solutions of (3.8.3) appropriately in phase space. First recall our decomposition (3.8.2):

$$\psi = \psi_H + \psi_R + \psi_{trap}.$$

We apply the decomposition (3.3.8) to $\psi_{trap} = \chi_{trap}\psi$:

$$\psi_{trap}(t, r, \theta, \phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\in\mathbb{Z}} \sum_{\ell>|m|} \chi_{trap}(r) R_{m\ell}^{(a\omega)}(r) S_{m\ell}^{(a\omega)}(\cos\theta) e^{im\phi} e^{-i\omega t}\, d\omega.$$

Let $\{\chi_n(r)\}_{n=1}^{\tilde{N}}$ be a smooth partition of unity subordinate to the cover $\left\{\tilde{A}_n\right\}_{n=1}^{\tilde{N}}$ such that

$$\text{supp } \chi_n(r) = \tilde{A}_n.$$

Define the cut-off functions as follows:

$$\alpha_n(\omega, m, \ell) = \chi_n(r_{m\ell}^{(a\omega)}).$$

We use these to localise in phase space by defining, for $(\omega, m, \Lambda) \in \mathcal{F}_\natural$:

$$\psi_n(t,r,\theta,\phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z}\\ \ell\geq|m|}} \alpha_n(\omega,m,\ell)\chi_{trap}(r)R_{m\ell}^{(a\omega)}(r)S_{m\ell}^{(a\omega)}(\cos\theta)e^{im\phi}e^{-i\omega t}\,d\omega. \tag{3.8.10}$$

To get the full solution of (3.8.3) upon summation, we need one more wave packet that takes into account $(\omega, m, \Lambda) \notin \mathcal{F}_\natural$. Define

$$\psi_0(t,r,\theta,\phi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{m\in\mathbb{Z}} \sum_{\ell>|m|} \alpha_0(\omega,m,\ell)\chi_{trap}(r)R_{m\ell}^{(a\omega)}(r)S_{m\ell}^{(a\omega)}(\cos\theta)e^{im\phi}e^{-i\omega t}\,d\omega$$

where

$$\alpha_0(\omega,m,\ell) = 1 - \sum_{n=1}^{\tilde{N}} \alpha_n(\omega,m,\ell).$$

Note that the estimate (3.5.11) for $\alpha_0(\omega,r,m,\ell)u_{m\ell}^{(a\omega)}$ will not degenerate as the phase-space support of this wave packet lies outside the trapping regime. Therefore the (NEB) statement in $\tilde{A}_{trap}$ for these modes is proved by the same argument as in §3.8.1, so

$$\int_{\Sigma_\tau} \mathbb{J}_\mu^N[\psi_0] \leq C_0 \int_{\Sigma_0} \mathbb{J}_\mu^N[\psi]n_{\Sigma_0}^\mu. \tag{3.8.11}$$

Remark Note that in this section, $R_{m\ell}^{(a\omega)}$ and $u_{m\ell}^{(a\omega)}$ denote the solutions of (3.3.10) and (3.3.12) respectively with $F_{m\ell}^{(a\omega)}$ and $H_{m\ell}^{(a\omega)}$ arising from the cut-off $\chi_{trap}(r)$ rather than the cut-off $\gamma(t^*)$.

Proposition 3.8.2. *Each wave packet ψ_n defined by* (3.8.10) *enjoys the following properties.*

1. *Each ψ_n is sufficiently integrable in the sense of* (3.3.1).
2. *The estimate* (3.5.11) *for $\alpha_n(\omega,m,\ell)u_{m\ell}^{(a\omega)}$ will degenerate only in $\tilde{A}_n$ as desired.*
3. *Every $(\omega,m,\Lambda) \in \mathcal{F}_\natural$ belongs to the phase-space support of some wave packet $\widehat{\psi_n}$.*

Proof. 1. This follows immediately from Plancherel and the assumption that ψ satisfies (3.3.1).

2. We have constructed α_n such that

$$\text{supp}\ \alpha_n(\omega,m,\ell) = \left\{(\omega,m,\ell) \mid r_{m\ell}^{(a\omega)} \in \tilde{A}_n\right\}.$$

3. This follows from the item above and the fact that $\left\{\tilde{A}_n\right\}_{n=1}^{\tilde{N}}$ covers $\tilde{A}_{trap}$. $\square$

3.8.4 (ILED) for wave packets

We first prove that the analogue of (ILED) holds for each wave packet.

Lemma 3.8.1. *There exists a positive constant C depending only on M, r_e, R_e and P such that for any wave packet ψ_n, $n \geq 1$, and any $\tau > 0$,*

$$\int_0^\tau \int_{\Sigma_t \cap [r_e^*, R_e^*]} \left[(\partial_{r^*}\psi_n)^2 + \psi_n^2 + \xi_n(r) \left((T\psi_n)^2 + |\nabla\!\!\!/ \psi_n|^2 \right) \right] dt^* \leq C \int_{\Sigma_0} \mathbb{J}_\mu^N[\psi] n_{\Sigma_0}^\mu, \tag{3.8.12}$$

where ξ_n degenerates only in $\tilde{A}_n$.

Proof. Note that $|\alpha_n(\omega, m, \ell)| \leq 1$ and $|\chi_{trap}| \leq 1$. Since $\chi_{trap}(r)$ and $\chi_n(r)$ are compactly supported smooth functions in r,

$$|\partial_r \chi_n|^2 + |\partial_r \chi_{trap}|^2 \leq \max_{1 \leq n \leq \tilde{N}} \sup_{r \in [r_{A_n}, R_{A_n}]} |\partial_r \chi_n|^2 + \sup_{r \in [r_e, R_e]} |\partial_r \chi_{trap}|^2 := C.$$

This constant is independent of ω, m and ℓ. Repeating the proof of Proposition 3.5.9 and recalling that $J_{m\ell}^{(a\omega)}$ is not supported in the trapping regime $\mathcal{F}_\natural$,

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \int_{r_e^*}^{R_e^*} \Bigg(\left| \frac{d}{dr^*} \left(\alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right) \right|^2 + \left| \alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right|^2 \\
&\qquad\qquad + (r - r_{m\ell}^{(a\omega)})^2 (\omega^2 + \Lambda) \left| \alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right|^2 \Bigg) dr^* d\omega \\
\leq\ & 4C \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \int_{r_e^*}^{R_e^*} \left| \frac{d}{dr^*} u_{m\ell}^{(a\omega)} \right|^2 + \left| u_{m\ell}^{(a\omega)} \right|^2 + (r - r_{m\ell}^{(a\omega)})^2 (\omega^2 + \Lambda) \left| u_{m\ell}^{(a\omega)} \right|^2 dr^* d\omega \\
\leq\ & \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \int_{\tilde{A}_{trap}} C_{m\ell}^{(a\omega)}(r^*) \mathrm{Re} \left[(u_{m\ell}^{(a\omega)})' \bar{H}_{m\ell}^{(a\omega)} \right] + D_{m\ell}^{(a\omega)}(r^*) \mathrm{Re} \left[u_{m\ell}^{(a\omega)} \bar{H}_{m\ell}^{(a\omega)} \right] dr^* d\omega \\
&+\ B \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell \geq |m|}} \int_{\tilde{A}_{trap}} \omega \mathrm{Im} \left[u_{m\ell}^{(a\omega)} \bar{H}_{m\ell}^{(a\omega)} \right] dr^* d\omega,
\end{aligned}$$

where we have absorbed constants,

$$H_{m\ell}^{(a\omega)}(r) := \frac{\Delta G_{m\ell}^{(a\omega)}(r)}{(r^2 + a^2)^{1/2}}$$

and

$$G_{m\ell}^{(a\omega)} := \int_{\mathbb{S}^2} \widehat{G_{trap}}(\omega, r, \theta, m) \cdot \overline{S_{m\ell}^{(a\omega)}(\cos\theta) e^{im\phi}} dV_{\mathbb{S}^2}.$$

By construction, $\chi_n(r_{m\ell}^{(a\omega)})$ is only supported in $\tilde{A}_n$ for $1 \le n \le \tilde{N}$. Hence $(r - r_{m\ell}^{(a\omega)})^2$ vanishes only in $\tilde{A}_n$ for $1 \le n \le \tilde{N}$ and $\chi_0 = 1$. From the identities in §3.3.3 and (3.8.10), we have for any $\tau > 0$

$$\begin{aligned}
&\int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell\ge|m|}} \int_{r_e^*}^{R_e^*} \left(\left| \frac{d}{dr^*} \left(\alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right) \right|^2 + \left| \alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right|^2 \right. \\
&\qquad\qquad \left. + (r - r_{m\ell}^{(a\omega)})^2 (\omega^2 + \Lambda) \left| \alpha_n \chi_{trap} u_{m\ell}^{(a\omega)} \right|^2 \right) dr^* d\omega \\
\ge\ & c \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \int_{r_+}^{R_e} \left[(\partial_{r^*}\psi_n)^2 + \psi_n^2 + \xi_n(r) \left((T\psi_n)^2 + (\nabla\!\!\!/\psi_n)^2 \right) \right] \ \rho^2 \sin\theta \ d\theta \ d\phi \ dr \ dt^* \\
\ge\ & c \int_0^{\tau} \int_{\Sigma_t \cap [r_e^*, R_e^*]} \left[(\partial_{r^*}\psi_n)^2 + \psi_n^2 + \xi_n(r) \left((T\psi_n)^2 + (\nabla\!\!\!/\psi_n)^2 \right) \right] \ dt^*,
\end{aligned}$$

where c is a positive constant depending on M, r_e and R_e and $\xi_n(r)$ is a strictly positive function outside of $\tilde{A}_n$ which vanishes on the physical space projection of $r_{m\ell}^{(a\omega)}$ for $r_{m\ell}^{(a\omega)} \in \tilde{A}_n$.

So we have

$$\begin{aligned}
& c \int_0^{\tau} \int_{\Sigma_t \cap [r_e^*, R_e^*]} \left[(\partial_{r^*}\psi_n)^2 + \psi_n^2 + \xi_n(r) \left((T\psi_n)^2 + (\nabla\!\!\!/\psi_n)^2 \right) \right] \ dt^* \\
\le\ & \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell\ge|m|}} \int_{\tilde{A}_{trap}} C_{m\ell}^{(a\omega)}(r^*) \mathrm{Re}\left[(u_{m\ell}^{(a\omega)})' \bar{H}_{m\ell}^{(a\omega)} \right] + D_{m\ell}^{(a\omega)}(r^*) \mathrm{Re}\left[u_{m\ell}^{(a\omega)} \bar{H}_{m\ell}^{(a\omega)} \right] \ dr^* d\omega \\
& + \ B \int_{-\infty}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell\ge|m|}} \int_{\tilde{A}_{trap}} \omega \mathrm{Im}\left[u_{m\ell}^{(a\omega)} \bar{H}_{m\ell}^{(a\omega)} \right] \ dr^* d\omega.
\end{aligned}$$

It thus remains to prove that the right hand side of this estimate is controlled by initial data. Recall that

$$\begin{aligned}
G_{trap} &= (\Box\chi_{trap})\psi + 2\nabla^{\mu}(\chi_{trap})\nabla_{\mu}\psi \\
&= \frac{1}{\rho^2 \sin\theta} \partial_r(\Delta \partial_r \chi_{trap})\psi + 2\frac{\Delta}{\rho^2} \partial_r(\chi_{trap})(\partial_r\psi). \qquad (3.8.13)
\end{aligned}$$

By the identities in §3.3.3,

$$B \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \sum_{\substack{m\in\mathbb{Z} \\ \ell\ge|m|}} \omega \mathrm{Im}\left[u_{m\ell}^{(a\omega)} \bar{H}_{m\ell}^{(a\omega)} \right] \ dr^* d\omega \ = \ B \int_{-\infty}^{\infty} \int_{r_+}^{\infty} \int_{\mathbb{S}^2} (T\psi)(G_{trap}) \ \Delta dr dg_{\mathbb{S}^2} dt$$

Observe that each term in G_{trap} contains factors of $(\partial_r\chi_{trap})$ or $(\partial_r^2\chi_{trap})$ and consequently

$$\zeta := \text{supp } G_{trap} \subset \{r_e - 2\epsilon \le r \le r_e - \epsilon\} \cup \{R_\natural + \epsilon \le r \le R_\natural + 2\epsilon\}.$$

Note that supp $H^{(a\omega)}_{m\ell} \subset \zeta$ as well. Recall from §3.6 that $C^{(a\omega)}_{m\ell}(r)$, $D^{(a\omega)}_{m\ell}(r)$ and $\chi_{trap}(r)$ are smooth functions, bounded uniformly with respect to ω, m, Λ, so by (3.8.8) there exists a positive frequency independent constant C such that

$$\begin{aligned}
&\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\ge|m|}}\int_{\tilde{A}_{trap}} C^{(a\omega)}_{m\ell}(r^*)\text{Re}\left[(u^{(a\omega)}_{m\ell})'\bar{H}^{(a\omega)}_{m\ell}\right] + D^{(a\omega)}_{m\ell}(r^*)\text{Re}\left[u^{(a\omega)}_{m\ell}\bar{H}^{(a\omega)}_{m\ell}\right]\, dr^* d\omega \\
&+\ B\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\ge|m|}}\int_{\tilde{A}_{trap}} \omega\text{Im}\left[u^{(a\omega)}_{m\ell}\bar{H}^{(a\omega)}_{m\ell}\right]\, dr^* d\omega \\
\le&\ C\int_{-\infty}^{\infty}\sum_{\substack{m\in\mathbb{Z}\\ \ell\ge|m|}}\int_{\zeta}\left|(u^{(a\omega)}_{m\ell})'\right|^2 + \left|u^{(a\omega)}_{m\ell}\right|^2 + \omega^2\left|u^{(a\omega)}_{m\ell}\right|^2 + \left|H^{(a\omega)}_{m\ell}\right|^2\, dr^* d\omega. \\
\le&\ C\int_{-\infty}^{\infty}\int_{\Sigma_t\cap\zeta}|\psi|^2 + |\partial_r\psi|^2 + |\partial_t\psi|^2\, dt.
\end{aligned}$$

Now ζ is disjoint from the trapping region, so we may apply (3.8.8) to obtain the result. □

3.8.5 (NEB) for wave packets

Proposition 3.8.3. *For each wave packet ψ_n, we have* (NEB). *That is, there exists a constant C_n, depending on M, r_e, R_e, P (from Proposition 3.3.1), λ_2, ω_1 and ω_2 such that*

$$\int_{\Sigma_\tau}\mathbb{J}^{Q_n}_\mu[\psi_n]n^\mu_{\Sigma_\tau} \le C_n\int_{\Sigma_0}\mathbb{J}^{Q_n}_\mu[\psi]n^\mu_{\Sigma_0}. \tag{3.8.14}$$

Proof. In light of (3.8.11), we need only consider the wave packets supported in $\mathcal{F}_\natural$, that is, ψ_n for $n \ge 1$.

Each wave packet ψ_n is sufficiently integrable in the sense of (3.3.1) so for each n there exists a dyadic sequence $\tau^{(n)}_j \to -\infty$ and a constant k_n such that

$$\int_{\Sigma_{\tau^{(n)}_j}}\mathbb{J}^{Q_n}_\mu[\psi_n]n^\mu_{\Sigma_{\tau^{(n)}_j}} \le \frac{-k_n}{\tau^{(n)}_j}. \tag{3.8.15}$$

Applying the energy identity for Q_n to ψ_n between a $\Sigma_{\tau_j^{(n)}}$ and Σ_τ for $1 \leq n \leq \tilde{N}$ we have

$$\begin{aligned}\int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_n] n^\mu_{\Sigma_\tau} &= \int_{\Sigma_{\tau_j^{(n)}}} \mathbb{J}^{Q_n}_\mu[\psi_n] n^\mu_{\Sigma_{\tau_j^{(n)}}} \\ &\quad - \int_{\tau_j^{(n)}}^{\tau} \int_{r_e-2\epsilon}^{R_e} \int_{\mathbb{S}^2} \left(\mathcal{E}^{Q_n}[\psi_n] + \mathbb{K}^{Q_n}[\psi_n]\right) \, \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt.\end{aligned}$$

By (3.8.15), taking $j \to \infty$, we have

$$\int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_n] n^\mu_{\Sigma_\tau} \leq \left| \int_{-\infty}^{\tau} \int_{r_e-2\epsilon}^{R_e} \int_{\mathbb{S}^2} \left(\mathcal{E}^{Q_n}[\psi_n] + \mathbb{K}^{Q_n}[\psi_n]\right) \, \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt \right|. \quad (3.8.16)$$

Since Q_n is a timelike Killing vector field in A_n and $Q_n = T$ for $r > R_\natural + 3\epsilon$, the bulk term $\mathbb{K}^{Q_n}[\psi_n]$ vanishes in these regions, so

$$\int_{-\infty}^{\tau} \int_{r \in [r_{A_n}, R_{A_n}] \cup \{r < r_e - 2\epsilon\} \cup \left\{r > R_\natural + 3\epsilon\right\}} \int_{\mathbb{S}^2} \mathbb{K}^{Q_n}[\psi_n] \; \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt = 0.$$

Moreover, (3.8.12) has no degeneracies outside of $\tilde{A}_n \subset A_n$, so

$$\int_{-\infty}^{\tau} \int_{\left\{r_e - 2\epsilon \leq r \leq R_\natural + 3\epsilon\right\} \setminus [r_{A_n}, R_{A_n}]} \int_{\mathbb{S}^2} \left|\mathbb{K}^{Q_n}[\psi_n] \; \rho^2 \sin\theta\right| \, dr \; d\theta \; d\phi \; dt \leq C \int_{\Sigma_0} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_0}.$$

It remains to control the error term. Recalling (3.8.13),

$$\begin{aligned}&\int_{-\infty}^{\tau} \int_{r_e-2\epsilon}^{R_e} \int_{\mathbb{S}^2} \mathcal{E}^{Q_n}[\psi_n] \; \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt \\ &= \int_{-\infty}^{\tau} \int_{r_e-2\epsilon}^{R_e} \int_{\mathbb{S}^2} (Q_n\psi_n)(G_{trap}) \; \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt \\ &= \int_{-\infty}^{\tau} \int_{r\in\zeta} \int_{\mathbb{S}^2} (Q_n\psi_n)(G_{trap}) \; \rho^2 \sin\theta \; dr \; d\theta \; d\phi \; dt \\ &\leq C(r_e, R_e, \epsilon, \chi_{trap}) \int_{-\infty}^{\tau} \int_{r\in\zeta} \int_{\mathbb{S}^2} \psi^2 + |\partial\psi|^2 \; dr \; d\theta \; d\phi \; dt,\end{aligned}$$

where

$$\zeta := \text{supp } G_{trap} \subset \{r_e - 2\epsilon \leq r \leq r_e - \epsilon\} \cup \{R_\natural + \epsilon \leq r \leq R_\natural + 2\epsilon\}.$$

Recall that the degeneracy of (3.8.12) is contained in the region $\{r_e < r < R_\natural\}$ which is disjoint from ζ. Therefore, the right hand side of the estimate above may now be controlled by applying (3.8.12). $\square$

3.8.6 (NEB) for the full solution

We now need to sum up the energies of the wave packets to carry this result over to the full solution ψ of (2.2.3).

Proposition 3.8.4. *For each vector field Q_n, $0 \leq n \leq \tilde{N}$ and $\tau \geq 0$, there exists a constant $c_n > 0$ such that*

$$\int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_\tau} \leq c_n \sum_{k=0}^{\tilde{N}} \int_{\Sigma_\tau} \mathbb{J}^{Q_k}_\mu[\psi_k] n^\mu_{\Sigma_\tau} + c_n \int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_H] n^\mu_{\Sigma_\tau} + c_n \int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_R] n^\mu_{\Sigma_\tau}. \tag{3.8.17}$$

Proof. From (2.2.7) we see that

$$\begin{aligned}
\int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_\tau} &= \int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\sum_{k=0}^{\tilde{N}} \psi_k + \psi_H + \psi_R] n^\mu_{\Sigma_\tau} \\
&\leq (c_n/2) \int_{\Sigma_\tau} \left| \partial \left(\sum_{k=0}^{\tilde{N}} \psi_k + \psi_H + \psi_R \right) \right|^2 \\
&\leq c_n \sum_{k=0}^{\tilde{N}} \int_{\Sigma_\tau} |\partial \psi_k|^2 + c_n \int_{\Sigma_\tau} |\partial \psi_H|^2 + c_n \int_{\Sigma_\tau} |\partial \psi_R|^2
\end{aligned}$$

which is the result modulo a constant which can be absorbed into c_n. □

We now complete the proof of (NEB). First apply Proposition 3.8.4. Then apply the (NEB)-type results Proposition 3.8.3, (3.8.4), (3.8.5) and (3.8.11).

$$\begin{aligned}
\int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_\tau} &\leq c_n \sum_{k=0}^{\tilde{N}} \int_{\Sigma_\tau} \mathbb{J}^{Q_k}_\mu[\psi_k] n^\mu_{\Sigma_\tau} + c_n \int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_H] n^\mu_{\Sigma_\tau} + c_n \int_{\Sigma_\tau} \mathbb{J}^{Q_n}_\mu[\psi_R] n^\mu_{\Sigma_\tau} \\
&\leq c_n (C_0 + C_H + C_R) \int_{\Sigma_0} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_0} + \sum_{n=1}^{\tilde{N}} C_n \int_{\Sigma_1} \mathbb{J}^{Q_n}_\mu[\psi] n^\mu_{\Sigma_1}.
\end{aligned}$$

This concludes the proof of Theorem 3.2.1.

Chapter 4

Quantitative mode stability for the wave equation on the Kerr–Newman spacetime

4.1 Introduction

In this chapter, mode stability results for solutions of the wave equation on a subextremal Kerr–Newman spacetime are proved. Both the qualitative results of the type proved in [Whi89] for the Kerr case and the extended, qualitative results of the type proved in [SR13] for the Kerr case are shown to hold in the Kerr–Newman case. In particular, the quantitative mode stability result is used to prove an energy estimate for low superradiant frequencies, required in the proof of Proposition 3.6.1.

As in the Kerr case, one of the major difficulties in understanding the wave equation on a Kerr–Newman background is that of *superradiance*, the fact that the conserved ∂_t energy is not positive definite and thus does not control the solution ψ. After an appropriate frequency localisation in the frequency parameters ω and m (corresponding to the Killing fields ∂_t and ∂_ϕ respectively), the superradiant frequencies are seen to be those satisfying

$$0 \leq m\omega \leq \frac{am^2}{2Mr_+ - Q^2}. \tag{4.1.1}$$

In particular, the ∂_t energy identity does not preclude finite-energy exponentially growing mode solutions (with explicit t, ϕ dependence $e^{-i\omega t}e^{im\phi}$) associated with the frequencies (4.1.1), with ω in the upper half-plane. The statement that such modes do not exist is known as *mode stability*. In the Kerr case, this has indeed been proven by Whiting in the celebrated [Whi89].

The proof of quantitative boundedness and decay for solutions of (1.2.1) in the Kerr case given in [DRSR14] in fact depended on a quantitative refinement of Whiting's [Whi89]. The necessary refinement was proved very recently by Shlapentokh-Rothman in [SR13] by first extending [Whi89] to exclude resonances on the real axis and then refining this qualitative statement to a quantitative estimate.[1]

Turning to the Kerr–Newman spacetimes, even the analogue of Whiting's mode stability is absent in the literature. In the present chapter, we will prove for these spacetimes both the qualitative mode stability results (in the upper half-plane and on the real axis) as well as the quantitative estimate in the spirit of [SR13]. In particular, the latter result is needed for the general boundedness and decay results presented in Chapter 3. The precise mode stability results are stated here in §4.5 and the estimate needed in in the proof of Proposition 3.6.1 is presented here in Theorem 4.8.2.

In the Kerr case, the crucial ingredients in the proof of mode stability given in [Whi89]

[1]In the case $|a| \ll M$, one need not appeal to Whiting's [Whi89] (or its refinement [SR13]) as the small parameter may be exploited to deal directly with superradiance. A boundedness result had been obtained for $|a| \ll M$ in [DR11b] followed by decay results in [AB09], [DR09] and [TT11]. For the situation in the extremal case $|a| = M$, see [Are12a] and [Are12b]. For the case where $\Lambda > 0$, see [Dya11] and for the $\Lambda < 0$ case, see [Gan12], [HS13a] and [HS13b].

and [SR13] are the remarkable transformation properties of the radial ODE satisfied by the modes. Miraculously, all the essential elements of this structure are preserved in passing from the Kerr to the Kerr–Newman solution. In particular, we show that the radial ODE can be represented as a confluent Heun equation (See §4.4). We then define the *Whiting transform* for $u(\omega, m, \lambda, r)$ with $Im(\omega) \geq 0$ (see (4.6.3) for the definition). The Whiting transform takes the solution u^* of a confluent Heun equation to $\tilde{u}$ which solves another confluent Heun equation with different coefficients (See Proposition 4.6.1). There are three pivotal facts about this transform:

(a) The potential in the confluent Heun equation satisfied by $\tilde{u}$ possesses certain positivity properties. (See Proposition 4.7.1.)

(b) $\tilde{u}$ has 'good' asymptotics near the horizon and near null infinity. (See Propositions 4.6.2 and 4.6.3.)

(c) For $\omega \neq 0$ on the real axis, the limit of u at the horizon is a positive multiple of the limit of $\tilde{u}$ at $r \to \infty$. (See Proposition 4.6.3.)

The statements above were shown to be true for the Kerr case in [Whi89] and [SR13]; there is no a priori reason why one would expect these properties to carry over to the Kerr–Newman case. It is thus a fortunate fact that the potential and Δ parameter for the Kerr–Newman case differ from those in the Kerr case in such a way that the conditions (a), (b) and (c) still hold. This is discussed further in §4.6.

4.2 Mode solutions of the wave equation

A general subextremal Kerr–Newman metric possesses only the two Killing fields ∂_t and ∂_ϕ. Nonetheless, Carter discovered in [Car68] that the wave equation (2.1.8) can be formally separated. This is related to the existence of an additional hidden symmetry. We use this to make the following definition:

Definition 4.2.1. *Let $(\mathcal{M}, g)$ be a subextremal Kerr–Newman spacetime. A smooth solution ψ of the wave equation* (2.1.8) *is called a* mode solution *if there exist $(\omega, m, \ell) \in \mathbb{C} \setminus \{0\} \times \mathbb{Z} \times \{\mathbb{Z} \,:\, \ell \geq |m|\}$ such that*

$$\psi(t, r, \theta, \phi) = R^{(a\omega)}_{m\ell}(r) S^{(a\omega)}_{m\ell}(\theta) e^{im\phi} e^{-i\omega t},$$

where

1. *$S^{(a\omega)}_{m\ell}$ solves the following Sturm-Liouville problem*

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dS^{(a\omega)}_{m\ell}}{d\theta}\right)-\left(\frac{m^2}{\sin^2\theta}-a^2\omega^2\cos^2\theta\right)S^{(a\omega)}_{m\ell}+\lambda^{(a\omega)}_{m\ell}S^{(a\omega)}_{m\ell}=0 \qquad (4.2.1)$$

with the boundary condition that

$$e^{im\phi}S^{(a\omega)}_{m\ell}(\theta) \text{ extends smoothly to } \mathbb{S}^2, \qquad (4.2.2)$$

with $S^{(a\omega)}_{m\ell}$ an eigenfunction with corresponding eigenvalue $\lambda^{(a\omega)}_{m\ell}$ of the angular ODE (4.2.1).[2]

2. *R solves the radial equation*

$$\left[\partial_r(\Delta\partial_r)-\omega^2\left(a^2-\frac{(a^2+r^2)^2}{\Delta}\right)+\frac{a^2m^2}{\Delta}-\frac{2am\omega(2Mr-Q^2)}{\Delta}-\lambda^{(a\omega)}_{m\ell}\right]R=0. \qquad (4.2.3)$$

3. *$R(r)(r-r_+)^{-\frac{i(am-(2Mr_+-Q^2)\omega)}{r_+-r_-}}$ is smooth at $r=r_+$.*[3]

4. *There exist constants $\{C_k\}_{k=0}^{\infty}$ such that for any $N\geq 1$,*

$$R(r^*)=\frac{e^{i\omega r^*}}{r}\sum_{k=0}^{N}C_k r^{-k}+O(r^{-N-2}),$$

for large r.[4]

The boundary conditions (4.2.2) and in points 3 and 4 above are uniquely determined by requiring that ψ extends smoothly to the horizon $\mathcal{H}^+$ and has finite energy along asymptotically flat hypersurfaces for $Im(\omega)>0$ and along hyperboloidal hypersurfaces for $Im(\omega)\leq 0$. See the discussion in [SR13, Appendix D] for details, cf. [Dya11] and [War13].

It is convenient to define

$$u^{(a\omega)}_{m\ell}(r^*)=\sqrt{r^2+a^2}R^{(a\omega)}_{m\ell}(r) \qquad (4.2.4)$$

[2] The Sturm–Liouville problem admits a set of eigenfunctions $\left\{S^{(a\omega)}_{m\ell}\right\}_{\ell=|m|}^{\infty}$ and real eigenvalues $\left\{\lambda^{(a\omega)}_{m\ell}\right\}_{\ell=|m|}^{\infty}$. The eigenfunctions $\left\{S^{(a\omega)}_{m\ell}\right\}$ are called "oblate spheroidal harmonics" and define an orthonormal basis for $L^2(\sin\theta d\theta)$.

[3] We will subsequently denote this as $R(r)\sim(r-r_+)^{\frac{i(am-(2Mr_+-Q^2)\omega)}{r_+-r_-}}$ at $r=r_+$.

[4] We will subsequently denote this as $R(r^*)\sim r^{-1}e^{i\omega r^*}$ as $r\to\infty$.

which satisfies the radial Carter ODE:

$$\frac{d^2}{(dr^*)^2}u_{m\ell}^{(a\omega)}(r^*) + \left(\omega^2 - V_{m\ell}^{(a\omega)}(r)\right)u_{m\ell}^{(a\omega)} = 0. \tag{4.2.5}$$

Note that even though $R_{m\ell}^{(a\omega)}$ is complex-valued, the potential $V_{m\ell}^{(a\omega)}$ is real (see (3.3.12) for more details).

We will often drop the indices ω, m, ℓ when there is no risk of confusion. We will also adopt the convention that u' denotes a derivative with respect to r^*.

4.3 The Wronskian

Through asymptotic analysis of (4.2.5), one can make the following definitions:

Definition 4.3.1. *Let $u_{hor}(r^*, \omega, m, \ell)$ be the unique function satisfying*

1. $u''_{hor} + (\omega^2 - V)u_{hor} = 0$.
2. $u_{hor} \sim (r - r_+)^{\frac{i(am-(2Mr_+-Q^2)\omega)}{r_+-r_-}}$ *as* $r^* \to -\infty$.
3. $\left|\left((r(r^*) - r_+)^{-\frac{i(am-(2Mr_+-Q^2)\omega)}{r_+-r_-}} u_{hor}\right)(-\infty)\right|^2 = 1$.

Definition 4.3.2. *Let $u_{out}(r^*, \omega, m, \ell)$ be the unique function satisfying*

1. $u''_{out} + (\omega^2 - V)u_{out} = 0$.
2. $u_{out} \sim e^{i\omega r^*}$ *as* $r^* \to \infty$.
3. $\left|\left(u_{out}e^{-i\omega r^*}\right)(\infty)\right|^2 = 1$.

One then defines the Wronskian

$$W(\omega, m, \ell) = u_{hor}(r^*)u'_{out}(r^*) - u'_{hor}(r^*)u_{out}(r^*). \tag{4.3.1}$$

The Wronskian can be evaluated at any fixed r^*. The Wronskian W will vanish if the solutions are linearly dependent. Then $W = 0$ implies $\left|W^{-1}\right| = \infty$. The quantitative mode stability result will be an explicit upper bound for $\left|W^{-1}\right|$, so that u_{out} and u_{hor} are linearly independent and any solution of the Carter ODE (4.2.5) can be expressed as a superposition of those solutions.

4.4 The inhomogeneous equation

In the proof of Theorem 4.5.1, we will consider the following inhomogeneous form of (4.2.3),

$$\left[\partial_r(\Delta\partial_r) - \omega^2\left(a^2 - \frac{(a^2+r^2)^2}{\Delta}\right) + \frac{a^2m^2}{\Delta} - \frac{2am\omega(2Mr-Q^2)}{\Delta} - \lambda_{m\ell}^{(a\omega)}\right] R_{m\ell}^{(a\omega)} = F, \tag{4.4.1}$$

where F is a compactly supported smooth function on (r_+,∞). The corresponding inhomogeneous version of (4.2.5) is then

$$\frac{d^2}{(dr^*)^2}u_{m\ell}^{(a\omega)}(r^*) + \left(\omega^2 - V_{m\ell}^{(a\omega)}(r)\right)u_{m\ell}^{(a\omega)} = H := \frac{\Delta F}{(r^2+a^2)^{1/2}}. \tag{4.4.2}$$

4.5 Statement of mode stability results

For a subextremal Kerr–Newman spacetime $(\mathcal{M},g)$, we have the following results.

Theorem 4.5.1 (Quantitative mode stability on the real axis). *Let*

$$\mathcal{F} \subset \{(\omega,m,\ell) \in \mathbb{R}\times\{\mathbb{Z}\times\mathbb{Z} \mid \ell \geq |m|\}\}$$

be a frequency range for which

$$C_{\mathcal{F}} := \sup_{(\omega,m,\ell)\in\mathcal{F}} \left(|\omega| + |\omega|^{-1} + |m| + \left|\lambda_{m\ell}^{(a\omega)}\right|\right) < \infty.$$

Then the Wronskian W given by (4.3.1) *satisfies*

$$\sup_{(\omega,m,\ell)\in\mathcal{F}} \left|W^{-1}\right| \leq G(C_{\mathcal{F}},a,Q,M).$$

where the function G can, in principle, be given explicitly.

In proving the quantitative result above, we will also obtain the following qualitative results.

Theorem 4.5.2 (Mode Stability on the real axis). *There exist no non-trivial mode solutions corresponding to $\omega \in \mathbb{R}\setminus\{0\}$.*

Theorem 4.5.3 (Mode Stability). *There exist no non-trivial mode solutions corresponding to $Im(\omega) > 0$.*

Theorem 4.5.3 is the analogue of Whiting's original mode stability result [Whi89]. Theorem 4.5.2 is the analogue of Shlapentokh-Rothman's extension of Whiting's mode stability result [Whi89] to the real axis. Theorem 4.5.1 is the quantitative refinement of Theorem 4.5.2 needed in Chapter 3 for the proof of linear stability of subextremal Kerr–Newman black holes.

Note that for non-superradiant frequencies ω, m, i.e. those outside of the range (4.1.1), Theorem 4.5.2 and Theorem 4.5.3 follow immediately from the energy identity (see [SR13, §1.5 & §1.6]). In what follows, we will not however make a distinction between superradiant and non-superradiant frequencies.

4.6 The Whiting transform

The problem with trying to derive energy estimates for the Carter ODE (4.2.5) is that the boundary condition at $r^* = -\infty$ may give a non-positive term due to superradiance. To deal with this, we will first cast (4.2.5) as a confluent Heun equation (4.6.2). Applying the Whiting transform (4.6.3) to (4.6.2), we will obtain a new confluent Heun equation (4.6.4) with different coefficients and boundary conditions that allow for a useful energy estimate.

4.6.1 The confluent Heun equation

We rescale R as follows. Let

$$u^* := e^{i\omega r}(r - r_-)^{-\eta}(r - r_+)^{-\xi} R(r) \tag{4.6.1}$$

where

$$\eta := -\frac{i\left(am - \omega\left(2Mr_- - Q^2\right)\right)}{r_+ - r_-} \quad \textit{and} \quad \xi := \frac{i\left(am - \omega\left(2Mr_+ - Q^2\right)\right)}{r_+ - r_-}.$$

Then u^* satisfies the following Confluent Heun equation:

$$(r - r_+)(r - r_-)\frac{d^2u^*}{dr^2} + \left(\gamma(r - r_+) + \delta(r - r_-) + p(r - r_+)(r - r_-)\right)\frac{du^*}{dr} \\ + \left(\alpha p(r - r_-) + \sigma\right)u^* = G \tag{4.6.2}$$

where

$$
\begin{aligned}
\gamma &:= 2\eta+1, \\
\delta &:= 2\xi+1, \\
p &:= -2i\omega, \\
\alpha &:= 1, \\
\sigma &:= 2am\omega - 2\omega r_- i - \lambda_{m\ell}^{(a\omega)} - a^2\omega^2 \\
\text{and } G &:= e^{i\omega r}(r-r_-)^{-\eta}(r-r_+)^{-\xi}F.
\end{aligned}
$$

This can be verified by a direct calculation, generalising the analogous computation in [Whi89].

Note that, as in the (subextremal) Kerr case, r_+ and r_- are distinct roots of Δ. If Δ had more roots, or if these roots were not distinct, the Carter ODE would lie in a different class of equations.

4.6.2 The transformed equation

We now generalise the Whiting transformation to the Kerr–Newman case.

Proposition 4.6.1. *Let $Im(\omega) \geq 0$, $\omega \neq 0$, and let R solve* (4.4.1) *with the boundary conditions of Definition 4.2.1. Define x^* analogously to r^* by*

$$\frac{dx^*}{dx} = \frac{x^2+a^2}{(x-r_+)(x-r_-)}, \qquad x^*(3M) = 0,$$

Then define $\tilde{u}$ by

$$\tilde{u}(x^*) := (x^2+a^2)^{1/2}(x-r_+)^{-2iM\omega}e^{-i\omega x} \times \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr \tag{4.6.3}$$

where

$$\eta := -\frac{i\left(am-\omega\left(2Mr_- - Q^2\right)\right)}{r_+-r_-} \quad \textit{and} \quad \xi := \frac{i\left(am-\omega\left(2Mr_+ - Q^2\right)\right)}{r_+-r_-}.$$

Then $\tilde{u}(x)$ is smooth on (r_+,∞) and satisfies the following confluent Heun equation:

$$\tilde{u}'' + \Phi\tilde{u} = \tilde{H}, \tag{4.6.4}$$

where primes denote derivatives with respect to x^*,

$$\begin{aligned}
\tilde{H}(x^*) &:= \frac{(x-r_+)(r-r_-)}{(x^2+a^2)^2}\tilde{G}(x), \\
\tilde{G}(x) &:= \frac{(x^2+a^2)^{1/2}}{(x-r_+)^{2iM\omega}}e^{-i\omega x}\int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}F(r)dr, \\
\Phi(x^*) &:= \frac{(x-r_-)(x-r_+)}{(a^2+x^2)^4}\left(\left(2x^2-a^2\right)(r_-r_+)-2Mx(x^2-2a^2)-3a^2x^2\right) \\
&\quad +\frac{(x-r_-)(x-r_+)}{(a^2+x^2)^2}\left(\frac{4am(x-M)\omega}{r_--r_+}-\lambda_{m\ell}^{(a\omega)}-a^2\omega^2\right. \\
&\quad \left.+\frac{8M^2(x-M)(x-r_-)\omega^2}{(r_--r_+)(r_+-x)}+\frac{(x-r_-)\left((r_+-r_-)(x-r_+)-4Q^2\right)\omega^2}{r_+-r_-}\right)
\end{aligned}$$

Proof. It turns out that the proof is a direct modification of the computations in [SR13, §4]. Let us remark on the fortuitous structure of the Kerr–Newman spacetimes that makes this so. We have already remarked in §4.6.1 that (4.6.2) is a confluent Heun equation and thus (at least formally) admits non-trivial transformations. The exponents η and ξ are obtained from indicial equation associated to (4.2.5). They are the unique exponents that give the correct asymptotics at r_+ and r_-.

The definitions of η, ξ, r_+ and r_- for the Kerr–Newman case differ from those in the Kerr case, but the potential $V_{m\ell}^{(a\omega)}$, the parameter Δ and the asymptotics of the solutions of mode solutions of (4.2.5), have the same structure. The convergence of the integral in (4.6.3) thus follows as in [SR13, §4]. □

Remark. The Whiting transform is a shifted, rescaled Fourier transform of a rescaled version of R. This fact will be crucial in showing that the vanishing of $\tilde{u}$ forces R to vanish.

4.6.3 Asymptotics of the transformed solution

The good asymptotic properties of $\tilde{u}$ (c.f. (b) and (c) of the introduction) are encapsulated in the following two propositions.

Proposition 4.6.2. *Let ω and $\tilde{u}$ be as in the statement Proposition 4.6.1. If $Im(\omega) > 0$ then*

1. $\tilde{u} = O\left((x-r_+)^{2MIm(\omega)}\right)$ *as* $x \to r_+$.
2. $\tilde{u}' = O\left((x-r_+)^{2MIm(\omega)}\right)$ *as* $x \to r_+$.
3. $\tilde{u} = O\left(e^{-Im(\omega)x^{1+2MIm(\omega)}}\right)$ *as* $x \to \infty$.

4. $\tilde{u}' = O\left(e^{-Im(\omega)x^{1+2MIm(\omega)}}\right)$ *as* $x \to \infty$.

Proposition 4.6.3. *Let* ω *and* $\tilde{u}$ *be as in the statement Proposition 4.6.1. If* $\omega \in \mathbb{R} \setminus \{0\}$ *then*

1. $\tilde{u}$ *and* $\tilde{u}'$ *are uniformly bounded.*
2. $|\tilde{u}(\infty)|^2 = \frac{(r_+ - r_-)^2 |\Gamma(2\xi+1)|^2}{4(2Mr_+ - Q^2)\omega^2} |u(-\infty)|^2$, *where* $\Gamma(z) := \int_0^\infty e^{-t} t^{z-1} dt$ *is the Gamma function.*
3. $\tilde{u}' - i\omega\tilde{u} = O\left(x^{-1}\right)$ *as* $x^* \to \infty$.
4. $\tilde{u}' + i\omega r_+^{-1}(r_+ - r_-)\tilde{u} = O\left(x - r_+\right)$ *as* $x^* \to -\infty$.

The proofs of these propositions are direct modifications of the computations in [SR13, §4].

For all the results above, except Proposition 4.6.3.2, the difference between the Kerr and Kerr–Newman case is encapsulated within the different definitions of r_+ and r_-.

Proposition 4.6.3.2 is exceptional in that we see an explicit difference from the Kerr case. This is due to the presence of $(2Mr_+ - Q^2)$ in the null generator of the Kerr–Newman horizon.

Proposition 4.6.3.2 is crucial in proving the quantitative result Theorem 4.5.1 as it provides a correspondence between the horizon asymptotics of the solution of the Carter ODE and the large r^* asymptotics of the transformed solution. This correspondence is what allows for the quantitative estimate of the horizon flux in terms of the inhomogeneity F (see the proof of Proposition 4.7.2).

We can now prove the qualitative Theorems 4.5.2 and 4.5.3.

4.7 Proofs of mode stability

4.7.1 Qualitative results

The final element of the structure necessary to prove mode stability for the Kerr–Newman spacetimes is the following positivity property (c.f. (a) of the introduction):

Proposition 4.7.1. *Under the conditions of Proposition 4.6.1,*

$$Im(\Phi\bar{\omega}) \geq 0.$$

If $\omega \in \mathbb{R} \setminus \{0\}$, *then* Φ *is real-valued.*

Proof. The second statement is clear from the definition of Φ. A (tedious) computation shows that

$$\begin{aligned}
&Im(\Phi\bar{\omega}) \\
= \; &\frac{(x-r_-)(x-r_+)}{(a^2+x^2)^2} Im\left((-\lambda_{m\ell}^{(a\omega)} - a^2\omega^2)\bar{\omega}\right) + \frac{(x-r_-)^2(x-r_+)^2\omega_I|\omega|^2}{(a^2+x^2)^2} \\
&+\frac{(x-r_-)^2(x-r_+)\omega_I|\omega|^2}{(a^2+x^2)^2}\frac{\left(8M^2(x-M)-4Q^2(x-r_+)+(r_+-r_-)(x-r_+)^2\right)}{(r_+-r_-)(x-r_+)} \\
&+\frac{(x-r_-)(x-r_+)}{(a^2+x^2)^4}(\omega_I)\left[x^2(r_+-a^2-Q^2)+r_-(x^2+a^2)(x-r_+)\right. \\
&\qquad\qquad\left. +2xa^2(x+r_--r_+)\right].
\end{aligned}$$

To see that $Im\left((-\lambda_{m\ell}^{(a\omega)} - a^2\omega^2)\bar{\omega}\right) \geq 0$, multiply (4.2.1) by $\overline{\omega S_{m\ell}^{(a\omega)}}\sin\theta$ and integrate by parts over $[0,\pi]$.

The positivity of the other terms follows from the following chain of inequalities

$$0 \leq r_- \leq M \leq r_+ \leq x$$

and the subextremal condition $a^2+Q^2<M^2$. □

We define the *microlocal energy current*

$$\tilde{Q}_T := Im(\tilde{u}'\overline{\omega\tilde{u}}).$$

Proof of Theorem 4.5.3 (Mode stability in the upper half-plane). Let $\omega = \omega_R + i\omega_I$ and $Im(\omega) = \omega_I > 0$ and consider a mode solution of (2.1.8) with $(u_{m\ell}^{(a\omega)}, S_{m\ell}^{(a\omega)}, \lambda_{m\ell}^{(a\omega)})$. Define $\tilde{u}$ to be the (4.6.3) of $u_{m\ell}^{(a\omega)}$. Then Proposition 4.6.2 implies that $\tilde{Q}_T(\pm\infty)=0$ so

$$0 = -\int_{-\infty}^{\infty}(\tilde{Q}_T)'dr^* = \int_{-\infty}^{\infty}\omega_I\left|\tilde{u}'\right|^2 + Im(\Phi\bar{\omega})|\tilde{u}|^2 dr^*$$

Since Proposition 4.7.1 guarantees that $Im(\Phi\bar{\omega}) \geq 0$, we conclude that, $\tilde{u}$, the Whiting transform of u vanishes. Hence

$$\tilde{R}(x) := \int_{r_+}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(x-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr = 0.$$

Extending R by 0, we see that the Fourier transform of $(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)$ is (up to a change of variable)

$$\hat{R}(z) := \int_{-\infty}^{\infty} e^{2i|\omega|^2(z-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr.$$

The function R is supported in $[0,\infty)$, so by the Paley–Wiener Theorem, $\hat{R}$ can be extended holomorphically into the upper half plane. Since $R=0$ for $x\in(-\infty,r_+)$ and $\tilde{R}=0$ for $x\in(r_+,\infty)$, $\hat{R}=0$ on the real line. We can therefore use the Schwartz reflection principle to extend $\hat{R}$ holomorphically to all of $\mathbb{C}$.

Furrthermore, the vanishing of $\tilde{R}$ on $x\in(r_+,\infty)$ implies that $\hat{R}=0$ on the line

$$\{z=\bar{\omega}(x-r_+)/(r_+-r_-)\mid x\in(r_+,\infty)\}.$$

The Identity Theorem for holomorphic functions then implies that $\hat{R}$ vanishes everywhere. This forces R to vanish everywhere, completing the proof. □

Lemma 4.7.1 (Unique continuation [SR13]). *Suppose that we have a solution $u(r^*)$: $(-\infty,\infty)\to\mathbb{C}$ to*

$$u''+(\omega^2-V)u=0$$

such that

1. $\omega\in\mathbb{R}\setminus\{0\}$,

2. *u is uniformly bounded and* $(|u'|^2+|u|^2)(\infty)=0$,

3. *V is real, uniformly bounded, $V=O(r^{-1})$ as $r\to\infty$ and $V'=O(r^{-2})$ as $r\to\infty$.*

Then u is identically 0.

Proof. This follows exactly as in [SR13, §6] □

Proof of Theorem 4.5.2 (Mode stability on the real axis). Let $\omega\in\mathbb{R}\setminus\{0\}$ and consider a mode solution of (2.1.8) with $(u^{(a\omega)}_{m\ell},S^{(a\omega)}_{m\ell},\lambda^{(a\omega)}_{m\ell})$. Define $\tilde{u}$ by (4.6.3). By Proposition 4.6.1, Φ is real, so $(\tilde{Q}_T)'=0$. Hence $\tilde{Q}_T(\infty)-Q_T(-\infty)=0$. The boundary conditions from Proposition 4.6.3 then imply that

$$\omega^2|\tilde{u}(\infty)|^2+\left|\tilde{u}'(\infty)\right|^2+\omega^2\frac{r_+-r_-}{r_+}|\tilde{u}(-\infty)|^2+\frac{r_+}{r_+-r_-}\left|\tilde{u}'(-\infty)\right|^2=0.$$

By Lemma 4.7.1, we conclude that $\tilde{u}$ vanishes.

Extending R by 0, we see that

$$\tilde{R}(y) := \int_{-\infty}^{\infty} e^{\frac{2i\omega}{r_+-r_-}(y-r_-)(r-r_-)}(r-r_-)^{\eta}(r-r_+)^{\xi}e^{-i\omega r}R(r)dr$$

vanishes for $\{y \in (r_+,\infty)\}$. Now repeating the closing argument of the proof of Theorem 4.5.3, we conclude that R must vanish everywhere. $\square$

4.7.2 Quantitative results

The strategy is to express $\tilde{u}$ in terms of the functions u_{out} and u_{hor} and W defined in §4.3 and obtain an an estimate for W^{-1} in terms of $u(-\infty)$. This quantity is then estimated using the ODE (4.4.2).

Proposition 4.7.2. *Define $\mathcal{F}$ as in Theorem 4.5.1. For $(\omega, m, \ell) \in \mathcal{F}$ let u solve* (4.4.2) *with $H(x^*)$ a smooth, compactly supported function. Then for sufficiently small $\epsilon > 0$, there exists a positive constant $C := C(\mathcal{F}, a, Q, M)$ such that*

$$|u(-\infty)|^2 \leq C\left(\epsilon^{-1}\int_{r_+}^{\infty}|F(r)|^2r^4dr\right).$$

Proof. Since $(\tilde{Q}_T)' = \omega Im(\tilde{H}\bar{u})$,

$$\int_{-\infty}^{\infty}\omega Im(\tilde{H}\bar{u})dr^* = \tilde{Q}_T(\infty) - \tilde{Q}_T(-\infty).$$

The boundary conditions from Proposition 4.6.3 imply that

$$\omega^2|\tilde{u}(\infty)|^2 + \left|\tilde{u}'(\infty)\right|^2 + \omega^2\frac{r_+-r_-}{r_+}|\tilde{u}(-\infty)|^2 + \frac{r_+}{r_+-r_-}\left|\tilde{u}'(-\infty)\right|^2 = \int_{-\infty}^{\infty}\omega Im(\tilde{H}\bar{u})dr^*.$$

So changing variables, applying the Plancherel identity and the Cauchy Schwarz inequality, we have

$$\omega^2|\tilde{u}(\infty)|^2 \leq \int_{-\infty}^{\infty}\omega Im(\tilde{H}\bar{u})dr^* \leq C\left(\epsilon^{-1}\int_{r_+}^{\infty}|F(r)|^2r^4dr + \epsilon\int_{r_+}^{\infty}|R(r)|^2dr\right).$$

Then by Proposition 4.6.3

$$|u(-\infty)|^2 = \frac{4\omega^2(2Mr_+-Q^2)}{|\Gamma(2\xi+1)|^2}|\tilde{u}(\infty)|^2 \leq C\left(\epsilon^{-1}\int_{r_+}^{\infty}|F(r)|^2r^4dr + \epsilon\int_{r_+}^{\infty}|R(r)|^2dr\right).$$

Finally,

$$\epsilon\int_{r_+}^{\infty}|R(r)|^2dr \leq C\int_{r_+}^{\infty}|F(r)|^2r^4dr,$$

by the same argument as found in [SR13, §5]. □

For the quantitative result, we construct mode solutions solutions to the Carter ODE from the Wronskian and apply the proposition above.

Lemma 4.7.2. *Let $H(x^*)$ be compactly supported. For any $(\omega, m, \ell) \in \mathcal{F}$ (where $\mathcal{F}$ is as defined in Theorem 4.5.1), the function*

$$u(r^*) = W(\omega, m, \ell)^{-1}\left(u_{out}(r^*)\int_{-\infty}^{r^*} u_{hor}(x^*)H(x^*)dx^* \right.$$
$$\left. + u_{hor}(r^*)\int_{r^*}^{\infty} u_{out}(x^*)H(x^*)dx^*\right)$$

satisfies

$$u'' + (\omega^2 - V)u = H$$

and the boundary conditions of a mode solution (see Definition 4.2.1).

Proof. This is verified by a direct calculation. □

Proof of Theorem 4.5.1 (Quantitative mode stability on the real axis). Take $\tilde{u}$ as defined in Lemma 4.7.2. Then

$$|u(-\infty)|^2 = \left|W^{-2}\right|\left|\int_{-\infty}^{\infty} u_{out}(x^*)H(x^*)dx^*\right|^2.$$

Rearranging this expression and applying Proposition 4.7.2 we find that

$$\left|W^{-2}\right| = \frac{|u(-\infty)|^2}{\left|\int_{-\infty}^{\infty} u_{out}(x^*)H(x^*)dx^*\right|^2} \leq C\frac{\int_{-\infty}^{\infty}\left|(r^2+a^2)^{1/2}\Delta^{-1}H(r^*)\right|^2 r^4 dr}{\left|\int_{-\infty}^{\infty} u_{out}(x^*)H(x^*)dx^*\right|^2}.$$

Note that by Proposition 4.6.3, for sufficiently large x, $\left|u_{out}(x) - e^{i\omega x}\right| < Cx^{-1}$ for an explicit C. Since W is independent of H we choose a compactly supported H for which the right hand side of the estimate above is finite. We thus have a quantitative estimate for $\left|W^{-2}\right|$. □

4.8 Application: Integrated local energy decay

We now apply Theorem 4.5.1 to prove Theorem 4.8.2, which provides a quantitative energy decay estimate for solutions of the wave equation (2.1.8) on subextremal Kerr–Newman spacetimes which are supported in a compact range of superradiant frequencies. This is the estimate appealed to in the proof of Proposition 3.6.1 to control the horizon term $|u_{m\ell}^{(a\omega)}(-\infty)|^2$ in the bounded superradiant frequency region.

We wish to apply Carter's separation to the solution of (2.1.8). In order to perform this separation, we must be able to take the Fourier transform in time. We therefore deal with solutions of (2.1.8). which belong to the following class of functions.

Definition 4.8.1. *A smooth function $f(t,r,\theta,\phi)$ is said to be admissible if for any multi-indices α, β s.t. $|\alpha| \geq 1$, $|\beta| \geq 0$, we have*

1. $\displaystyle\int_{r>r_0}\int_{\mathbb{S}^2} |\partial^\alpha f|^2|_{t=0} r^2 \sin\theta dr\, d\theta\, d\phi < \infty$ *for sufficiently large r_0.*
2. $\displaystyle\int_0^\infty \left|\partial^\beta f\right|^2 dt < \infty$ *for any $(r,\theta,\phi) \in (r_+,\infty)\times\mathbb{S}^2$.*
3. $\displaystyle\int_0^\infty \int_K \left|\partial^\beta f\right|^2 \sin\theta\, dr\, d\theta\, d\phi\, dt < \infty$ *for any compact $K \in (r_+,\infty)\times\mathbb{S}^2$.*

For an admissible function f we also define

$$|\partial f|^2 := |(\partial_t + \partial_{r^*})f|^2 + \Delta|(\partial_t - \partial_{r^*})f|^2 + r^{-2}\left(\sin^{-2}\theta|\partial_\phi f|^2 + |\partial_\theta f|^2\right).^5 \tag{4.8.1}$$

The main application of Theorem 4.5.1 in §3.6 is to admissible solutions ψ of (2.1.8) which are cut off as follows.

Definition 4.8.2. *Let Σ_0 be a spacelike hyperboloidal hypersurface connecting the horizon $\mathcal{H}^+$ and future null infinity. Let Σ_1 be the time 1 image of Σ_0 under the flow generated by ∂_t. Then define a smooth cut-off γ which is identically 0 in the past of Σ_0 and identically 1 in the future of Σ_1. We define $\psi_{\text{✂}} := \gamma\psi$, which satisfies the inhomogeneous wave equation*

$$\Box_g \psi_{\text{✂}} = F, \quad \textit{where} \quad F = (\Box\gamma)\psi + 2\nabla^\mu\gamma\nabla_\mu\psi. \tag{4.8.2}$$

Proposition 4.8.1 (Carter's separation)**.** *Admissible solutions f of (2.1.8) and (4.8.2) can be expressed as*

$$f(t,r,\theta,\phi) = \overbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \underbrace{\sum_{m,\ell\geq|m|} R_{m\ell}^{(a\omega)}(r)\cdot S_{m\ell}^{(a\omega)}(\cos\theta)e^{im\phi}}_{\text{Oblate spheroidal expansion}} e^{-i\omega t}}^{\text{Fourier transform}}\, d\omega. \tag{4.8.3}$$

The function $R_{m\ell}^{(a\omega)}$ corresponding to $f = \psi$ solves (4.2.3). The function $R_{m\ell}^{(a\omega)}$ corresponding to $f = \psi_{\text{✂}}$ satisfies the inhomogeneous equation (4.4.1) with $F = F_{m\ell}^{(a\omega)}$, the Fourier

[5] The apparent degeneration of this energy as $r \to \infty$ is due to the hyperboloidal nature of Σ_0. The term $\Delta|(\partial_t - \partial_{r^*})f|^2$ converges to the transversal derivative at the horizon.

transform of F projected to the oblate spheroidal harmonic corresponding to $\lambda_{m\ell}^{(a\omega)}$. The rescaled function $u_{m\ell}^{(a\omega)}$ satisfies (4.4.2) with $H = H_{m\ell}^{(a\omega)} := \Delta(r^2+a^2)^{-1/2}F_{m\ell}^{(a\omega)}$, where this equality is to be understood in the sense of $L^2_{\omega\in\mathcal{B}}\ell^2_{m,\ell\in\mathcal{C}}$. Note moreover that this H is not compactly supported.

Proof. See §3.3.3. □

Theorem 4.8.2. *Let $\psi_{✂}$ be an admissible solution of (4.8.2) and let $\mathcal{B}\subset\mathbb{R}$ and*

$$\mathcal{C}\subset\{(m,\ell)\in\mathbb{Z}\times\mathbb{Z}\mid \ell\geq|m|\}$$

such that

$$C_{\mathcal{B}} := \sup_{\omega\in\mathcal{B}}\left(|\omega|+|\omega|^{-1}\right)<\infty \text{ and } C_{\mathcal{C}} := \sup_{m,\ell\in\mathcal{C}}\left(|m|+\left|\lambda_{m\ell}^{(a\omega)}\right|\right)<\infty.$$

There exists a constant $K := K(r_0, r_1, C_{\mathcal{B}}, C_{\mathcal{C}}, a, Q, M)$ such that

$$\begin{aligned}&\int_{\mathcal{B}}\sum_{m,\ell\in\mathcal{C}}\left(\left(\left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2+\left|u_{m\ell}^{(a\omega)}(\infty)\right|^2\right)+\int_{r_0}^{r_1}\left|\partial_{r^*}u_{m\ell}^{(a\omega)}\right|^2+\left|u_{m\ell}^{(a\omega)}\right|^2 dr^*\right)\,d\omega\\&\leq K\int_{\Sigma_0}|\partial\psi|^2,\end{aligned}\tag{4.8.4}$$

where $|\partial\psi|^2$ is defined by (4.8.1), $u_{m\ell}^{(a\omega)} = \sqrt{r^2+a^2}R_{m\ell}^{(a\omega)}$ and each $R_{m\ell}^{(a\omega)}$ solves (4.4.1) for $\omega\in\mathcal{B}$ and $(m,\ell)\in\mathcal{C}$.

Proof. For u satisfying the hypotheses of the theorem, we have for any $r^*\in(-\infty,\infty)$,

$$\begin{aligned}u(r^*) &= W(\omega,m,\ell)^{-1}\left(u_{out}(r^*)\int_{-\infty}^{r^*}u_{hor}(x^*)H(x^*)dx^*\right.\\&\qquad\left.+u_{hor}(r^*)\int_{r^*}^{\infty}u_{out}(x^*)H(x^*)dx^*\right),\end{aligned}\tag{4.8.5}$$

$$\begin{aligned}u'(r^*) &= W(\omega,m,\ell)^{-1}\left(u'_{out}(r^*)\int_{-\infty}^{r^*}u_{hor}(x^*)H(x^*)dx^*\right.\\&\qquad\left.+u'_{hor}(r^*)\int_{r^*}^{\infty}u_{out}(x^*)H(x^*)dx^*\right),\end{aligned}\tag{4.8.6}$$

where the inequalities above hold in the sense of $L^2_{\omega\in\mathcal{B}}\ell^2_{m,\ell\in\mathcal{C}}$ (see [SR13, §3] for the full derivation of this representation).[6]

[6] Roughly speaking, this is the converse of Lemma 4.7.2.

By the construction of u_{hor} and u_{out}, there exists a positive $K := K(C_{\mathcal{B}}, C_{\mathcal{C}}, a, Q, M)$ such that

$$\sup_{r^*\in\mathbb{R},\omega\in\mathcal{B},(m,\ell)\in\mathcal{C}} (|u_{hor}| + |u_{out}|) < K < \infty, \tag{4.8.7}$$

Evaluating (4.8.5) at $r^* = -\infty$ and taking (4.8.7) into account,

$$\int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} \left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2 d\omega \leq K \limsup_{r^*\to-\infty} \int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} W^{-2}\left|\int_{r^*}^{\infty} u_{out}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 d\omega. \tag{4.8.8}$$

For the term $|u(\infty)|^2$ we apply the microlocal energy current:

$$\begin{aligned}
\omega^2\left|u_{m\ell}^{(a\omega)}(\infty)\right|^2 &= Q_T(\infty) = Q_T(-\infty) + \int_{-\infty}^{\infty} (Q_T)' dr^* \\
&= \omega(am - (2Mr_+ - Q^2)\omega)\left|u_{m\ell}^{(a\omega)}(-\infty)\right|^2 + \omega \int_{-\infty}^{\infty} Im(H_{m\ell}^{(a\omega)} \bar{u}_{m\ell}^{(a\omega)}) dr^*
\end{aligned}$$

So by (4.8.8),

$$\begin{aligned}
\int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} \left|u_{m\ell}^{(a\omega)}(\infty)\right|^2 d\omega &\leq K \int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} W^{-2}\left|\int_{-\infty}^{\infty} u_{out}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 d\omega \\
&\quad + \int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} \omega \int_{-\infty}^{\infty} Im(H_{m\ell}^{(a\omega)} \bar{u}_{m\ell}^{(a\omega)}) dr^*\, d\omega.
\end{aligned} \tag{4.8.9}$$

For the integral term, we begin by taking R_1 much larger than r_1 and applying (4.8.5):

$$\begin{aligned}
&\int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} \sup_{r^*\in(r_0,r_1)} \left|u_{m\ell}^{(a\omega)}\right|^2 d\omega \\
\leq\ & K \int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} W^{-2} \left(\sup_{r^*\in[r_0,r_1]} \left|\int_{-\infty}^{r^*} u_{hor}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 \right. \\
&+ \sup_{r^*\in[r_0,r_1]} \left|\int_{r^*}^{R_1} u_{out}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 \\
&\left. + \left|\int_{R_1}^{\infty} u_{out}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 \right) d\omega \\
\leq\ & K \int_{\mathcal{B}} \sum_{m,\ell\in\mathcal{C}} W^{-2} \left(\int_{r_+}^{R_1} |F|^2 dr + \left|\int_{R_1}^{\infty} u_{out}(x^*)H_{m\ell}^{(a\omega)}(x^*)dx^*\right|^2 \right) d\omega.
\end{aligned}$$

This estimate may be integrated over (r_0, r_1) to obtain

$$\int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} \int_{r_0}^{r_1} \left| u_{m\ell}^{(a\omega)} \right|^2 d\omega \leq K \int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} W^{-2} \int_{r_+}^{R_1} |F|^2 dr + K \int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} W^{-2} \left| \int_{R_1}^{\infty} u_{out}(x^*) H_{m\ell}^{(a\omega)}(x^*) dx^* \right|^2 d\omega. \quad (4.8.10)$$

The same argument, with (4.8.5) replaced with (4.8.6) yields

$$\int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} \int_{r_0}^{r_1} \left| (u_{m\ell}^{(a\omega)})' \right|^2 d\omega \leq K \int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} W^{-2} \int_{r_+}^{R_1} |F|^2 dr + K \int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} W^{-2} \left| \int_{R_1}^{\infty} u_{out}(x^*) H_{m\ell}^{(a\omega)}(x^*) dx^* \right|^2 d\omega. \quad (4.8.11)$$

Collecting (4.8.8), (4.8.9), (4.8.10) and (4.8.11) and applying Theorem 4.5.1 to control W^{-2}, we have

$$\begin{aligned} &\int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} \left(\left(\left| u_{m\ell}^{(a\omega)}(-\infty) \right|^2 + \left| u_{m\ell}^{(a\omega)}(\infty) \right|^2 \right) + \int_{r_0}^{r_1} \left| \partial_{r^*} u_{m\ell}^{(a\omega)} \right|^2 + \left| u_{m\ell}^{(a\omega)} \right|^2 dr^* \right) d\omega \\ \leq \quad & KG \int_{\mathcal{B}} \sum_{m,\ell \in \mathcal{C}} \left[\left| \int_{R_1}^{\infty} u_{out}(x^*) H_{m\ell}^{(a\omega)}(x^*) dx^* \right|^2 + \int_{r_+}^{R_1} |F|^2 dr \right. \\ & \left. + \omega \int_{-\infty}^{\infty} Im(H_{m\ell}^{(a\omega)} \bar{u}_{m\ell}^{(a\omega)}) dr^* \right] d\omega. \end{aligned}$$

It remains to control the right hand side of this estimate by $\int_{\Sigma_0} \mathbb{J}^N[\psi] \cdot n_{\Sigma_0}$. The control of the first term is achieved using the proof of [SR13, Lemma 3.3]. The remaining terms are controlled using the argument presented in §3.6. □

Remark We can replace the hyperboloidal hypersurface Σ_0 with an asymptotically flat hypersurface in Theorem 4.8.2 as follows. Let Σ_0^* be an asymptotically flat hypersurface that agrees with Σ_0 for $\{r \leq R\}$ and which lies in the past of Σ_0. Choosing R large enough that T is timelike in $\{r \leq R\}$, applying the T energy estimate immediately implies that

$$\int_{\Sigma_0} |\partial \psi|^2 \leq C \int_{\Sigma_0^*} \left| \nabla_{g_{\Sigma_0^*}} \psi \right|^2 + \left| n_{\Sigma_0^*} \psi \right|^2,$$

so we can then replace the right hand side of (4.8.4) by this integral over an asymptotically flat hypersurface.

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